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# BRAESS'S PARADOXES IN DYNAMIC TRAFFIC ASSIGNMENT WITH SIMULTANEOUS DEPARTURE TIME AND ROUTE CHOICES

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In this paper, we investigate some new dynamic phenomena of Braess's paradox in transportation networks when both the departure time and route choices are considered simultaneously. The classical Wardropian user equilibrium principle is used to characterize route choice behavior, and the traditional bottleneck models with deterministic queues are employed to describe user's departure time choice behavior. New Braess's paradoxes with dynamic user response have been found in respect to the point queue and physical queue assumptions, and the mechanisms of such paradoxes are analyzed in details. Furthermore, it is shown in a numerical example that a paradox may occur in network when physical queue assumption is made, but the paradox may not be found in network if the point queue assumption is adopted in the proposed dynamic traffic assignment model. Due to the significant difference between point queue and physical queue assumptions, spatial dimension of queues has to be considered carefully in dynamic traffic assignment models. Finally, some traffic management measures are suggested and examined to resolve the potential paradoxes, such as link closing, ramp metering, dynamic road pricing, lane partitioning, and adaptive traffic control etc.

KEYWORDS: Braess's paradox, simultaneous departure time and route choices, dynamic user equilibrium, bottleneck, queue spillover

## 1. INTRODUCTION

Braess's paradox is a counterintuitive fact that adding new links or expanding capacities of existing links in a network can increase travel costs due to selfish routing. Following the pioneer work of Braess (1968, 2005), paradox of network expansion has attracted substantial interest among transportation scholars. Hallefjord et al. (1994) revisited Braess's paradox in networks with elastic demand from the perspective of economic welfare. Later, Pas and Principio (1997) and Penchina (1997) showed the dependence of paradoxes on link performance functions and travel demand levels. Yang (1997) demonstrated that adding capacities to some links might result in Braess's paradoxes, and applied sensitivity analysis method to identify potential links that might have a paradoxical effect. Yang and Bell (1998) introduced the concept of network capacity paradox that states that the addition of a new link might actually reduce the reserve capacity of the network. Recently, Valiant and Roughgarden (2006) showed that Braess's paradox occurs with high probability in natural large random networks, and hence paradox is not only a theoretical curiosity but also a common real-world phenomenon. Furthermore, empirical evidences of Braess's paradox in real-life networks have been recorded in the literature (Fisk and Pallottino, 1981; Murchland, 1970; Kolata, 1990).

With the development of techniques in dynamic traffic assignment, more and more researchers are concerned about the paradoxes under dynamic user equilibrium. Dynamic user equilibrium represents a situation that no traveler can reduce individual

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travel cost by unilaterally changing departure time and/or travel route. For instance, Arnott et al. (1993) discussed a paradox arising from user's departure time choice behavior on a Y-shaped highway corridor. Arnott and Small (1994) showed that the capacity expansion of a link might not reduce the overall travel cost of the network. Daganzo (1996, 1998) introduced several interesting paradoxes in networks with physical queues, where adding new ramps or expanding the capacity of existing bottlenecks may result in the abandonment of alternative routes due to the queue spillovers. Lin (1999) also introduced paradoxes caused by route choice competition in networks with physical queues, but departure time competition was not considered. Akamatsu and Kuwahara (1999) analyzed a capacity increasing paradox for a dynamic traffic assignment with departure time choice. Akamatsu (2000) investigated capacity paradoxes in one-to-many networks with point queues considering dynamic user equilibrium. Akamatsu and Heydecker (2003) provided general paradox conditions on some specific saturated networks with route choice.

Although Braess's paradox has been widely studied in the literature, even in dynamic setting. However, in the previous studies on Braess's paradoxes, only either route choice or departure time choice is considered in modeling traveler's travel behavior. And researchers set rather strict assumptions in the networks to be analyzed, such as with saturated queues in all links, without route overlap, and restricted ODs etc. Furthermore, a comprehensive analysis of resolutions to paradoxes is still lacking. To deepen the analysis of paradoxes in network expansion, we revisit this interesting topic in a more realistic situation, namely in general networks with simultaneous choices of traveling routes and departure times, and potential paradox resolutions are provided.

In the paper, it is assumed that travelers from the same OD pairs will face the same travel cost in choosing traveling routes and departure times, and no one can reduce travel cost by unilaterally altering travel decisions (Huang and Lam, 2002; Zhang et al., 2007; Szeto and Lo, 2004, 2006). Furthermore, individuals have a common preferred arrival time $t^*$ at the destination, and early and/or late arrival will be penalized. In our analysis, the travel cost from the origin to the destination departing at time $t$ and using route $i$ is

$$c^i(t) = \alpha \cdot T^i(t) + \beta \cdot \max\left\{0, t^* - t - T^i(t)\right\} + \gamma \cdot \max\left\{0, t + T^i(t) - t^*\right\}, \quad (1)$$

where $T^i(t)$ is the travel time of route $i$ for departure time $t$, $\beta$ is the schedule penalty for a unit time of early arrival, and $\gamma$ is that for a unit time of late arrival. For rationality, the relation $\gamma > \alpha > \beta$ holds according to the estimates by Small (1982).

Moreover, we assume that each link in networks is made up of two parts, namely the main body on which traffic is freely flowing with travel time $t^0$ and a bottleneck at the downstream end with service capacity $s$. Traffic can leave the link freely and incur no delay until its flow exceeds the capacity; and once flow exceeds capacity the following deterministic queuing process is used to estimate the delay

$$\frac{\mathrm{d}D(t)}{\mathrm{d}t} = \begin{cases} r\left(t-t^0\right) - s, & \text{if } r\left(t-t^0\right) > s \text{ or } D(t) > 0, \\ 0, & \text{if } r\left(t-t^0\right) \le s \text{ and } D(t) = 0, \end{cases} \quad (2)$$

where $D(t)$ is the queue length at the bottleneck at time $t$, and $r(t-t^0)$ is the flow rate entering the link at time $t-t^0$. Therefore, the travel time on link $i$ for a vehicle entering the link at time $t-t_i^0$ is $t_i^0 + D(t)/s_i$ .

In the study, two types of queues are considered, namely point queues and physical queues. In the case of point queues, vehicles have no physical length and hence are vertically piled up, and thus a link can hold infinite vehicles. Whereas in the case of physical queues, vehicles have certain spatial length and a link can only store a finite number of vehicles, therefore we have to add a constraint of $D(t) \le l_i$ on each link. Here $l_i$ is the holding capacity of link *i*.

The network equilibrium can be solved in two steps: first solving for the departure rate on each route given the number of travelers, and second determining the number of travelers on each route by equating travel cost of active routes. Please refer to Arnott et al. (1990) for the details of the two-step method. Note that in the equilibrium models adopted in this paper, departure time and route choices are considered simultaneously, which differs from the dynamic route choice models (Morikawa and Miwa, 2006) where an exogenous time dependent OD matrix is required (Cheung et al., 2006; Wong and Tong, 1998). In this paper, we use simple network examples to show various paradoxical phenomena with both point queues and physical queues, and propose some methods to resolve these paradoxes.

## 2. PARADOX OF NETWORK EXPANSION WITH POINT QUEUES

### *2.1 A paradox example of network expansion with point queues*

In this section, we use an example to show the paradoxical phenomenon of network expansion considering point queue bottleneck congestion.

#### *2.1.1 Example 1*

Figure 1a is a network with 3 nodes and 2 links. Node C represents the city center, and node A and node B are two residential areas. There are $n_a = 5.0\times10^4$ residents (group A) living in the area represented by node A, and $n_b = 6.0\times10^4$ residents (group B) living in the area represented by node B. In the morning, they all travel to the city center (node C). The two origins are connected to the city center via two routes. Link 1 (route 1) is a highway with longer distance but larger capacity, and link 2 (route 2) is an expressway with shorter distance but smaller capacity. The free-flow travel time of link 1 is $t_1^0 = 50$ min, subject to a bottleneck with capacity $s_1$ = 600veh/min. A *vertical* queue will develop if arrival rate exceeds service capacity. The free-flow travel time of link 2 is $t_2^0 = 0$ min, and its bottleneck capacity is $s_2 = 400$ veh/min. Both groups of individuals are assumed to have a common preferred arrival time at work (e.g. their official work start time), $t^*$ = 8:00 a.m. To simplify calculation, assume that late arrival is not allowed, or the cost per unit time of arrival late is $\gamma$ = \$∞/min. The cost per unit time of arrival early is $\beta$ = \$0.2/min, and the cost per unit in-vehicle travel time is $\alpha$ = \$0.4/min.

Traffic patterns on the two routes can be obtained by solving two independent classical bottleneck models. The equilibrium flow pattern in the network is shown in Figure 2. Line *AB* is the departure curve (from home) of group B, and *CD* is the arrival curve (for work) of group B. Similarly lines *EF* and *EG* are the departure (from home) and arrival (for work) curves of group A. Individual travel cost is \$25 for group A, and \$40 for group B. Therefore, total system travel cost of both groups is $\$3.65\times10^6$.

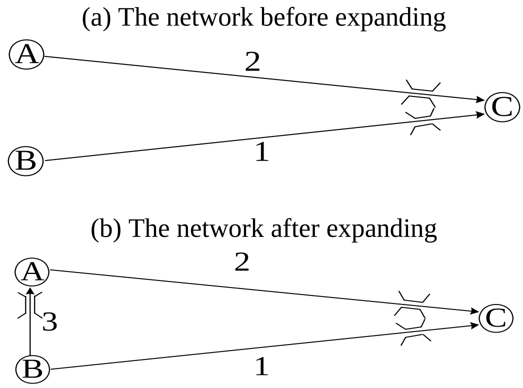


FIGURE 1: The networks used in Example 1

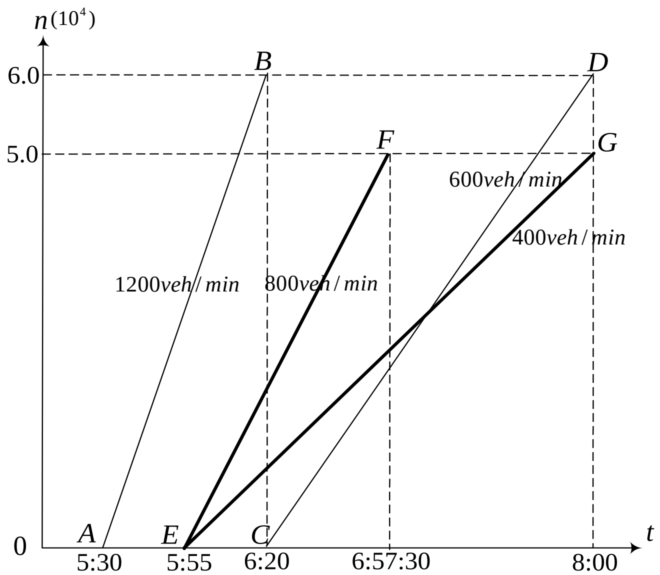


FIGURE 2: The dynamic traffic pattern on the network of Figure 1a

Now we look at the traffic equilibrium of the new network shown in Figure 1b, which is expanded from the original network in Figure 1a by constructing a new link from node B to node A. If the travel cost of group B is less than the travel cost of group A before adding link 3, or equivalently $\alpha t_2^0 + \beta n_a / s_2 \geq \alpha t_1^0 + \beta n_b / s_1$, no one will shift from route 1 to route 3 (link 3 and link 2) after link 3 is added. On the contrary, if $\alpha t_2^0 + \beta n_a / s_2 < \alpha t_1^0 + \beta n_b / s_1$, certainly some travelers shift from route 1 to route 3 after the construction of link 3. In fact, the latter case is a modified version of the network shown in Figure 1 of Arnott et al. (1993). In this case, group B is divided into two subgroups, namely group B1 who uses the original route 1 and group B2 who switches to route 3. Therefore, there are two levels of equilibria in the network. The first level concerns the departure time choice of travelers, which ensures that travelers who share the same route have equal travel cost. The second level is the spatial equilibrium between route 1 and route 3, which results in equal travel cost of groups B1 and B2.

Firstly, we look at the first level of equilibrium by treating the number of travelers in group B2 $x$ as an exogenous parameter. Now we divide the network into two parts, namely the lower part (route 1) used by group B1, and the upper part (route 2 and route 3) used by group B2 and group A. The departure pattern of route 1 can be solved by a typical bottleneck model given capacity $s_1$ and demand $n_b - x$, and the individual travel cost is $c_1 = \alpha t_1^0 + \beta(n_b - x)/s_1$. The upper part is equivalent to the network used in Arnott et al. (1993). In the example network of Arnott et al. (1993), bottleneck 1 has infinite capacity, so the origin of group 1 can be moved to the intersection point. Therefore the network in Arnott et al. (1993) is equivalent to the upper corridor of the network discussed here. By adding a constant into the travel cost of each link, results of Arnott et al. (1993) can be used directly. In equilibrium, the individual travel cost of route 3 is given by

$$c_3 = \begin{cases} \alpha t_2^0 + \alpha t_3^0 + \beta\left(x + n_a\right)/s_2, & \text{if } x \le \bar{x} \text{ (Case B)}, \\ \alpha t_2^0 + \alpha t_3^0 + \beta\left(x + n_a\right)/s_2, & \text{if } x > \bar{x} \text{ and } s_3 > s_2 \text{ (Case A1)}, \\ \alpha t_2^0 + \alpha t_3^0 + \beta\left(\dfrac{x}{s_3} + \dfrac{\beta n_a}{\alpha\left(s_2 - s_3\right) + \beta s_3}\right), & \text{if } x > \bar{x} \text{ and } s_3 \le s_2 \text{ (Case A2)}, \end{cases} \tag{3}$$

where $\bar{x} = n_a / \left\{\left(s_2\alpha / s_3(\alpha - \beta)\right) - 1\right\}$. The conditions $x \le \bar{x}$, $x > \bar{x}$ and $s_3 > s_2$, and $x > \bar{x}$ and $s_3 \le s_2$ correspond to the cases B, A1 and A2 in Arnott et al. (1993) respectively.

And, the individual travel cost of route 2 is given by

$$c_2 = \begin{cases} \alpha t_2^0 + \beta\left(x + n_a\right)/s_2, & \text{if } x \le \bar{x} \text{ (Case B)}, \\ \alpha t_2^0 + \alpha\left(\dfrac{x + n_a}{s_2} - \dfrac{x}{s_3}\right), & \text{if } x > \bar{x} \text{ and } s_3 > s_2 \text{ (Case A1)}, \\ \alpha t_2^0 + \beta\left(\dfrac{\alpha n_a}{\alpha\left(s_2 - s_3\right) + \beta s_3}\right), & \text{if } x > \bar{x} \text{ and } s_3 \le s_2 \text{ (Case A2)}. \end{cases} \tag{4}$$

Now the spatial equilibrium between route 1 and route 3 can be solved by equating $c_1$ and $c_3$, and the solution $x^*$ is the equilibrium demand of route 3. Since $c_1$ is a decreasing function of $x$, $c_3$ is an increasing function of $x$, and $\alpha t_2^0 + \beta n_a / s_2 < \alpha t_1^0 + \beta n_b / s_1$, there is one and only one solution of $x^*$. Substituting $x^*$ into the above equations yields system travel cost $n_a c_2(x^*) + (n_b - x^*)c_1(x^*) + x^* c_3(x^*)$.

Now we suppose link 3 has a free-flow travel time $t_3^0 = 0$ min and service capacity $s_3 = 200$ veh/min. After adding the link, some travelers in group B shift to link 2 by passing through link 3. In equilibrium, total numbers of travelers using links 1, 2 and 3 are $v_1 = 4.25 \times 10^4$ veh, $v_2 = 6.75 \times 10^4$ veh, and $v_3 = 1.75 \times 10^4$ veh respectively. Individual travel cost of group A is \$33.33, and that of group B is \$34.17. And system travel cost increases to $\$3.717 \times 10^6$.

This example tells us such a fact that adding a link to a network with intention to improve the transportation system, may lead to a deterioration of the network. Thus it can be regarded as a dynamic version of Braess's paradox. What we have learned from this example is that expanding network without fully considering the reaction of travelers is very dangerous. Note that compared with the static networks, the paradox observed in this example considers a more general setting, such as work start time, physical bottleneck capacity, unpunctuality penalty, and real time queuing delay etc. It is also revealed that if a network is inappropriately expanded, paradox is still unavoidable even travelers have more relaxed decisions, e.g. relatively free choice of departure time.

*2.2 When does paradox occur in Example 1?*

Firstly, whether paradox occurs or not in the above example depends on the capacity of bottleneck 3. Using the analytical methods described in the above subsection, traffic equilibrium in the network can be calculated straightforwardly given varying capacities of bottleneck 3, and the results are depicted in Figure 3. Clearly when $s_3$ increases, travel cost of group A increases and that of group B decreases. Interestingly, when $s_3$ increases from zero, the system travel cost decreases first and then increases. Since plotting Figure 3 is a one-variable problem, it is very easy to obtain the optimal capacity by trial-and-error method. As a result, the minimal system travel cost is \$3.605×10$^6$ when $s_3 = 78$ veh/min. When 78 veh/min $\geq s_3 \geq$ 0 veh/min, a larger $s_3$ makes more travelers shift from link 1 to links 2 and 3, which results in the reduction of system cost since the decrease of total travel cost of group B can over offset the increase of total travel cost of group A. At the optimal capacity of $s_3 = 78$ veh/min, the decrease of group B's total travel cost is equal to the increase of group A's total travel cost when a marginal traveler shifts from route 1 to route 3. When $s_3 \geq 78$ veh/min, further growth of capacity $s_3$ results in the increase of system cost since the decrease of total travel cost of group B can not offset the increase of total travel cost of group A. Again by trial-and-error method, it is found that when $s_3 = 156$ veh/min the system cost increases to the value equivalent to the situation of $s_3 = 0$ veh/min (or no link 3), in other words 156 veh/min is a breakeven capacity. Moreover, when $s_3$ increases to 215 veh/min, the queue in link 3 disappears and all travelers in both groups have equal travel cost, therefore further increase in $s_3$ does not affect traffic pattern in the network. From the above analysis, when $s_3 \leq 156$ veh/min there will be no paradox, and $s_3 = 78$ veh/min is the optimal design.

Secondly, the free flow travel time of link 3 $t_3^0$ is another factor that may induce paradox. For any given $t_3^0$, we can also figure out how equilibrium traffic pattern changes with $s_3$. In other words, given a $t_3^0$ we can determine some important values accordingly, such as optimal capacity, breakeven capacity, minimal system cost etc. The results are shown in Figure 4. Clearly as $t_3^0$ increases, both optimal capacity and breakeven capacity decrease, but the minimal system cost under optimal capacity increases. Furthermore, as $t_3^0$ reaches 30 min, paradox will surely happen for any positive capacity.

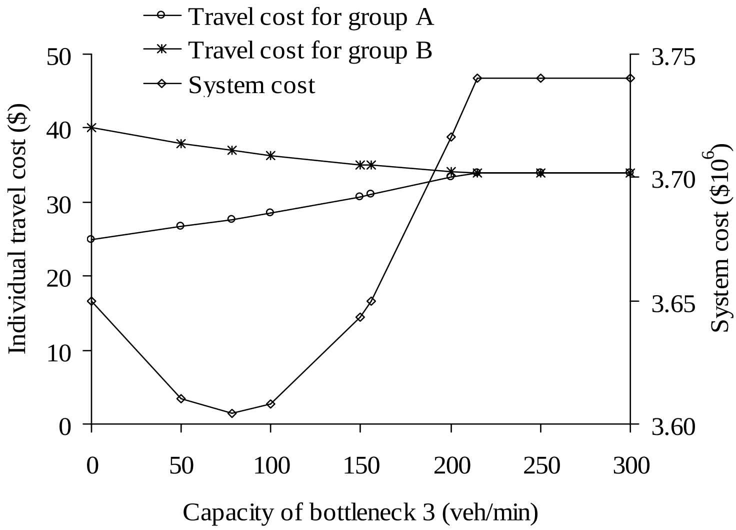


FIGURE 3: Variation of travel costs with the capacity of bottleneck 3

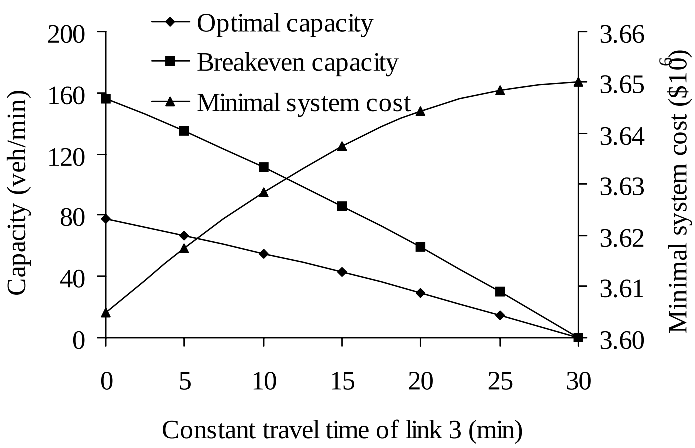


FIGURE 4: Variations of critical capacities and minimal system cost with the free-flow travel time of link 3

Thirdly, whether paradox happens or not also depends on the demands of both traveler groups. By changing both $n_a$ and $n_b$ from $4\times10^4$ to $8\times10^4$, the percentage reduction of system cost after introducing the new link 3 is shown in Figure 5 for each combination of $n_a$ and $n_b$ (given $t_3^0$=0 min and $s_3$=200 veh/min). In the area closed by the two zero lines and the axes, the system cost increases after introducing link 3. Therefore, when the combination of demands is within this area, paradox will surely occur. Interestingly, there is a triangle area in the up-left part of the chart, in which system cost decreases after introducing link 3. This implies when demand of group B is much greater than that of group A, the whole network will benefit from link 3 by sacrificing group A that is only a minor portion of the system. Moreover, there is also a triangle area in the bottom-right part of the chart, in which system cost maintains unchanged after introducing link 3. In this area, adding link 3 brings no effect to the network since link 2 becomes severely congested and no one will shift to the new route from link 1.

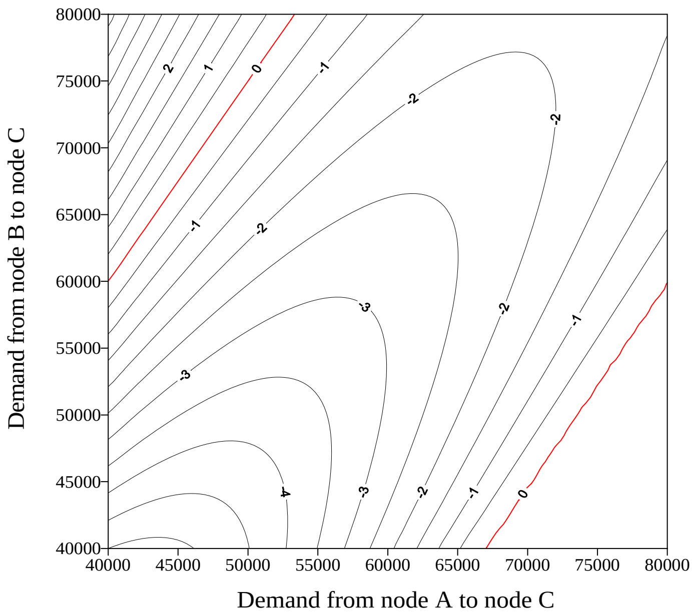


FIGURE 5: Relative (percent) reduction of system cost after adding link 3 given various OD demands

## 3. PARADOX OF NETWORK EXPANSION WITH PHYSICAL QUEUES

### *3.1 A paradox example of network expansion with physical queues*

In this section, we use the classical Braess's network to show the existence of paradox in network equilibrium with physical queues.

#### *3.1.1 Example 2*

Figure 6a is a network with four nodes and four links, which was used for illustrating the classical Braess's paradox in static user equilibrium. Free-flow travel times on links 1, 2 and 3 are zero, and that on link 4 is 10min. Capacities of bottlenecks 1, 2, 3 and 4 are 20veh/min, 24veh/min, 24veh/min, and 20veh/min respectively. Suppose links 1, 3 and 4 can hold infinite vehicles, and link 2 can hold 120 vehicles. Totally 2400 residents live in node 1, and everyday they need to work at node 4 starting from 8:00am. Parameters $\alpha$ = \$1.0/min, $\beta$ = \$0.5/min and $\gamma$ = \$2.0/min. Here physical queues will build up if arrival rates exceed capacities of the bottlenecks.

Clearly bottlenecks 1 and 4 are critical bottlenecks that determine the traffic conditions. Equilibrium traffic patterns on both routes are shown in Figure 7. Traffic pattern of route

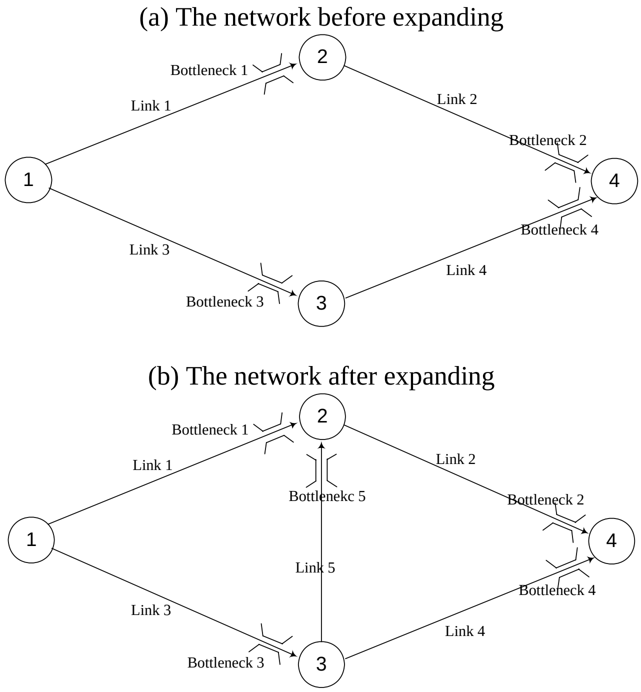


FIGURE 6: The networks used in Example 2

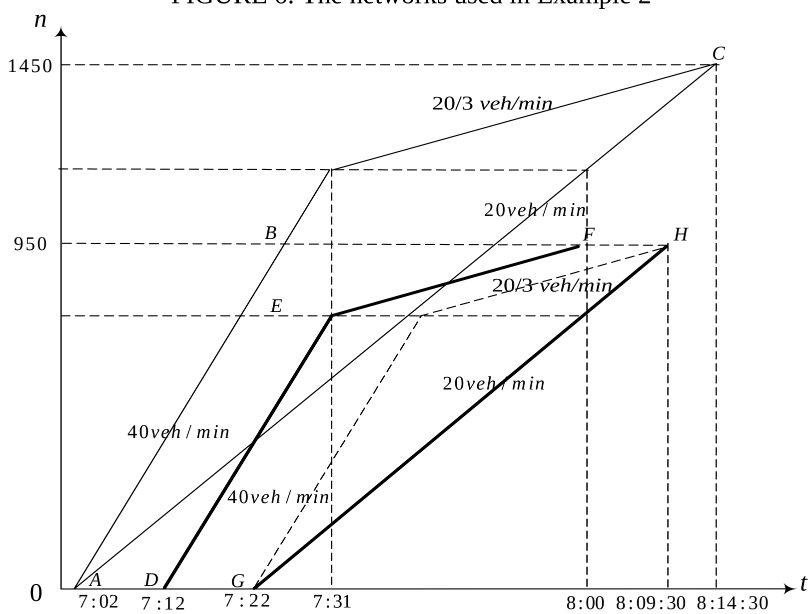


FIGURE 7: Dynamic traffic pattern on the network in Figure 6a

1 (links 1 and 2) is depicted by the regular lines, whereas that of route 2 (links 3 and 4) is depicted by the bold lines. Curves *ABC* and *AC* are the departure and arrival curves for route 1. Similarly, curves *DEF* and *GH* are the departure and arrival curves for route 2.

In equilibrium the number of travelers using route 1 is $N^1$ = 1450, and the number of travelers using route 2 is $N^2$ = 950. Travel costs on both routes are equal, and given by \$29. In this case, total travel cost in the network is $\$6.960\times10^4$.

Now a new link is constructed from node 3 to node 2, as shown in Figure 6b. The bottleneck on link 5 has a capacity of $s_5$ = 20 veh/min, and it is located at the downstream end of link 5. Link 5 has no free-flow travel time and can hold infinite vehicles. Now besides the previous two routes, there is a new route (route 3 with links 3, 5 and 2) available to travel. From the setting of unpunctuality parameters, the ratios of route inflow rate to route outflow rate for early and late arrivals are 2 and 1/3 respectively (Arnott et al., 1990). Since route 1 and route 3 are shorter than route 2 in terms of free-flow travel time, travelers will use route 1 and route 3 first. When route 1 and route 3 are in use, the outflow rates for both routes are equal and given by 12 veh/min, since the inflow rates to bottleneck 2 from both routes are equal and given by the upstream bottleneck capacity 20 veh/min. So bottlenecks 1, 2, 3 and 5 are operating in full capacities, traffic arrival rates for these bottlenecks are 24 veh/min, 40 veh/min, 24 veh/min, and 24 veh/min respectively. Therefore, physical queues are developing at the entrances of bottlenecks 1, 2 and 5. After 7.5 minutes, the queue in link 2 reaches 120 veh, and link 2 is filled up with the queue. From then on, the over-spilled vehicles wait in link 1 and link 5, and increase the queuing growth rate on both links. It is reasonable to assume that the route inflow and outflow rates keep unchanged after link 2 is filled up. After 10 minutes from the first departure, travel time of new departures in routes 1 and 3 catches up with the free-flow travel time of route 2, and route 2 starts to be used. To keep a user equilibrium, total waiting time on link 5 and link 2 maintains being the free-flow travel time of link 4 (i.e. 10 min), therefore the inflow rate of link 5 reduces to 12 veh/min. It is easy to see that the outflow rate of route 2 is also 12 veh/min, since the summation of outflow rates of route 2 and route 3 is bounded by the capacity of bottleneck 3. Due to the difference of free-flow travel time of the routes, the last traveler of route 2 arrives at the destination 5 minutes earlier than the last travelers of route 1 and route 3. With the analytical bottleneck models, traffic patterns on these routes can be easily established as shown in Figure 8. In the figure, *JKL* and *JL* represent departure and arrival curves of route 1 and route 3, *MNS* and *PQ* represent departure and arrival curves of route 2 respectively. With these route flows, the queuing curves of the bottlenecks are shown in Figure 9. Since bottleneck 4 has been always operating under capacity, there is no queuing curve for bottleneck 4 in Figure 9. In equilibrium, 900 travelers use route 1, 900 travelers use route 3, and 600 travelers use route 2.

For travelers using routes 1 and 3, the travel time corresponding to each departure (or arrival) is determined by the horizontal distance between curves *JKL* and *JL* in Figure 8. For travelers using route 2, the travel time is calculated by the horizontal distance between curves *MNS* and *PQ*. As a result, the en route travel time (in unit of minute) or en route travel cost (in unit of \$) corresponding to an arrival time *t*, is

$$T(t)=\begin{cases} 30-0.5(8:00-t), & t\in[7:00,\ 8:00] \text{ for routes 1 \& 3,} \\ & t\in[7:20,\ 8:00] \text{ for route 2,} \\ 30-2(t-8:00), & t\in[8:00,\ 8:15] \text{ for routes 1 \& 3,} \\ & t\in[8:00,\ 8:10] \text{ for route 2.} \end{cases}$$

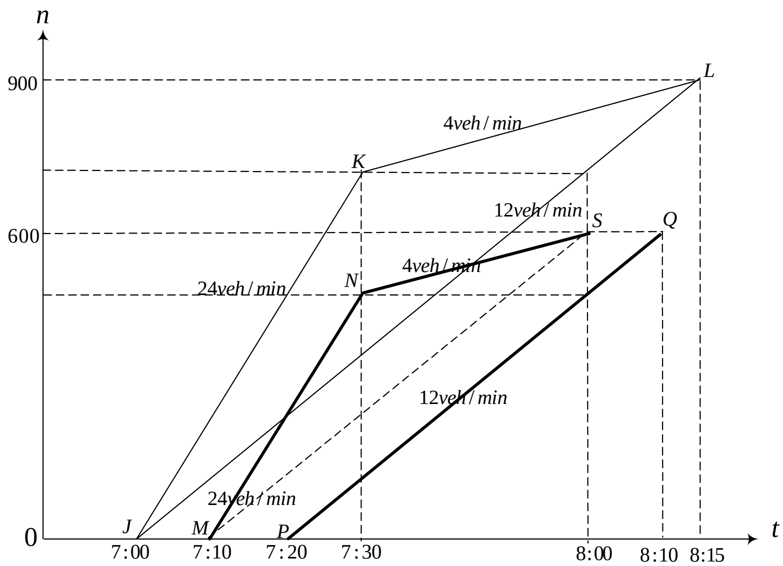


FIGURE 8: Dynamic traffic pattern on the network in Figure 6b

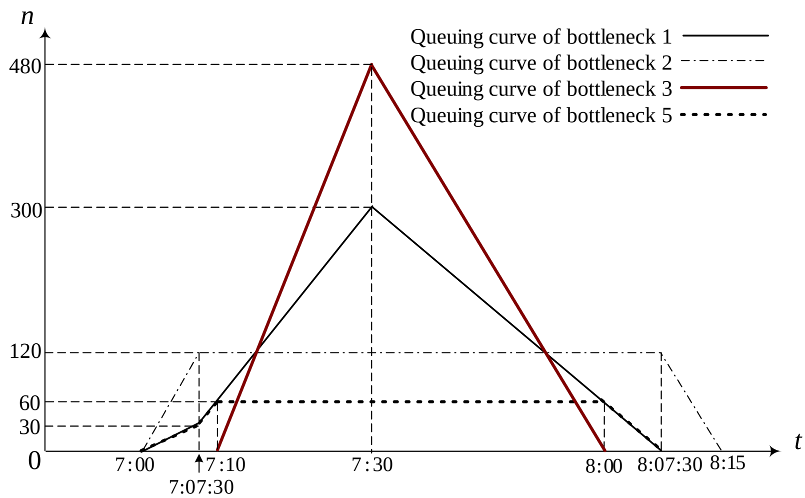


FIGURE 9: Time dependent queuing length of bottlenecks

The unpunctuality cost (in unit of $) corresponding to an arrival time $t$ is

$$U(t)=\begin{cases} 0.5(8:00-t), & t\in[7:00,\ 8:00] \text{ for routes 1 \& 3,} \\ & t\in[7:20,\ 8:00] \text{ for route 2,} \\ 2(t-8:00), & t\in[8:00,\ 8:15] \text{ for routes 1 \& 3,} \\ & t\in[8:00,\ 8:10] \text{ for route 2.} \end{cases}$$

Therefore, in this case individual travel cost is $30, and system travel cost becomes $\$7.2\times10^4$. Therefore, adding a link to the network leads to an increase of system travel cost by $\$2.4\times10^3$.

Note that the mechanism of this paradox is different from the case in Example 1. The paradox here is caused by the reduction of total throughput of the network after link 2 is filled up with traffic. Using the new route increases the throughput of bottleneck 2 by 4 veh/min, but reduces the throughput of bottleneck 4 by 8 veh/min. Thus the rush hour is prolonged by the new link, which results in an increase of travel cost. On the contrary, if the storage capacity of link 2 is infinite or point queue is assumed, paradox will not happen here. With point queue assumption, 10 minutes after the first departures of route 1 and route 3, total waiting time in bottlenecks 2 and 5 becomes 10 min, then inflow rate to link 5 decreases to 12, and route 2 starts to be used with an outflow rate of 12 veh/min. After 5 more minutes, inflow rate to link 5 further reduces to 4 veh/min, since the waiting time in bottleneck 2 reaches 10 min, and the outflow rate of route 2 increases to 20 veh/min. In this case, individual travel cost is $26.73. Therefore, if link 2 has sufficient storage capacity, adding link 5 can reduce individual travel cost by $2.27.

It is worth noting that the reason of the paradox in this example is different from that in Daganzo (1998). In Daganzo (1998) the paradox is caused by the abandonment of an alternative route when queue spills back to the diverge node. But in this paper, the paradox is caused by the reduction of total throughput of the network after link 2 is reaching to its capacity. The difference is that in this example all the alternative paths are still chosen by travelers although queue spillback occurs. Moreover, a more general setting for the departure time choice is incorporated in the proposed model.

### *3.2 When does paradox happen in Example 2?*

Certainly the capacity of bottleneck 5 ($s_5$) is important regarding whether paradox occurs or not. When $s_5 \leq 4$ veh/min, the total throughput of route 1 and route 3 increases and the throughput of route 2 does not change, so no paradox occurs. When $s_5 > 4$ veh/min, increasing the capacity attracts more flow to route 3, and thus results in a smaller throughput of route 2. So the travel cost decreases with the increase of $s_5$. When $s_5 = 14.55$ veh/min, individual travel cost is $29. In summary, to prevent paradox from occurring, $s_5$ cannot exceed 14.55 veh/min.

## 4. RESOLUTIONS TO PARADOXES UNDER DYNAMIC USER EQUILIBRIUM

### *4.1 Link closing*

Obviously paradoxes are caused by adding new links unreasonably, therefore the simplest way to resolve paradoxes is to close the improperly designed links. However, in most situations link closing may not be a positive strategy, since it will be criticized because of the waste of the new links.

### *4.2 Ramp metering*

Instead of fully closing the paradoxical links, ramp metering can make the best use of the capacity of these links. As shown in Section 2.2, the paradox in Example 1 is substantially affected by the capacity of bottleneck 3. By adjusting the capacity to the optimal one shown in Figure 4, a maximal system efficiency can be achieved. And in Example 2, the paradox caused by queue spillover can also be eliminated by metering link 5 to keep its capacity below 14.55 veh/min.

*4.3 Road pricing*

Road pricing has long been viewed as an efficient means to improve the performance of transportation network (Zhang and Yang, 2004; Yang and Zhang, 2002; Zhang et al., 2007; Zhang et al., 2008). It is also effective in resolving paradoxes in the literature of static traffic assignment (Yang, 1997). In the following, we show that road pricing is also applicable in resolving paradoxes in dynamic user equilibrium.

First, road pricing can function as link closing. The reason is simple. If we charge a sufficiently large toll on the paradoxical links, these links will never be used. In Example 1, if we charge a toll greater than $15, link 3 will never be used. And in Example 2, a toll greater than $10 suffices to close link 5.

Second, paradoxes that are caused by queue spillovers can be resolved by dynamic tolls, since bottleneck queues can be totally removed by charging time-varying road tolls. This policy is very desirable for the following two reasons. On one hand, by eliminating queue spillovers the network throughput is optimized. On the other hand, deadweight waiting costs can be eliminated by replacing queuing time with equivalent time-varying road tolls. In Example 2, the dynamic tolls shown in Figure 10 can lead to the arrival pattern equivalent to the case of $s_5$ = 4 veh/min, and meanwhile eliminate all queues in the network. This toll pattern ends up with an individual travel cost of $26.36, and a revenue of $\$2.755\times10^4$. If the revenue is equally returned to all travelers, individual travel cost further reduces to $14.89.

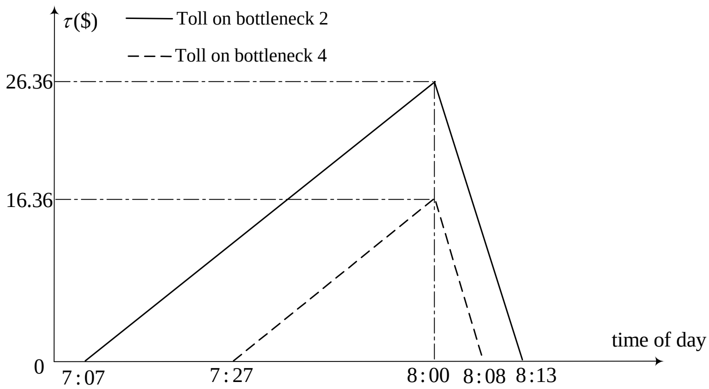

FIGURE 10: Time-varying bottleneck tolls

*4.4 Lane partitioning*

The paradox caused by the queue spillover in Example 2 can also be resolved by partitioning link 3 into two sets of lanes. As shown in Figure 11, link 3 is split into link 3′ and link 3″. Link 3′ sends flow to link 5, and link 3″ sends flow to link 4. Furthermore, to avoid confliction of the two distinct traffic streams, lane changing between the two lane groups is not allowed. The total capacity of bottlenecks 3′ and 3″ is equal to the original capacity of bottleneck 3. As long as the capacity of bottleneck 3′ is set below 14.55 veh/min, no paradox occurs.

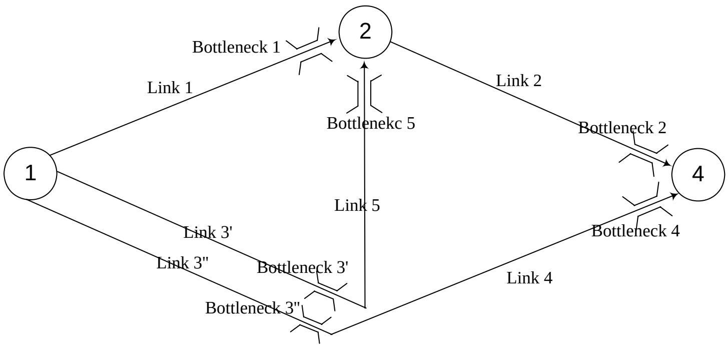


FIGURE 11: New network after lane partitioning and lane-change prohibition

*4.5 Adaptive traffic control*

In Example 2, if detectors have been installed at the upstream end of link 2, adaptive traffic management can be implemented to avoid paradox. If a queue is detected at the upstream end of link 2, flow rate in link 5 can be adjusted to an appropriate level by various traffic-responsive measures, such as dynamic metering, dynamic signal timing, and dynamic traffic diversion etc. This strategy can avoid queue spillovers even temporal traffic demand varies randomly.

## 5. CONCLUDING REMARKS

In this paper, we examined the well-known paradoxical phenomena in network design problems while the dynamic traffic assignment was adopted for modeling the classical Braess's paradox in the time-dependent dimension. The traditional bottleneck models were used to describe traveler's departure time choice behavior, and Wardropian user equilibrium principle was used for modeling route choice behavior.

Paradox was found when the point queue model was applied to a simple network with two origins and one destination connected by two routes, and each origin stands for a specific travel demand. One route is a highway with a larger capacity but longer distance, and the other route is an expressway with smaller capacity but shorter distance. If we build a ramp to link the two origins, system travel cost increases unexpectedly. The reason is that some travelers who previously used the highway would shift to the expressway, and the travel time saving in the highway cannot offset the travel time increase at the expressway. This example shows that we have to be very cautious when linking two different road systems. And it has practical implications since we often see such paradoxes in real life engineering. For example, in some cities, there are highway corridors surrounding urban area, and these highways are mainly designed for serving the cross-city traffic. Experiences often tell us that it is somehow dangerous to build convenient shortcuts linking such highway system and urban streets. The reason is that these shortcuts will bring cross-city traffic into urban streets and make the urban road network more congested. Therefore, the cross-city traffic should often deliberately be kept somewhat away from the intra-city traffic.

On the classical Braess's paradox network, we investigated the paradox problem in the dynamic traffic assignment model with physical queues. We showed that paradox might happen if we build a ramp from one route to the other route. Total throughput of the network is reduced after the downstream link of the ramp is filled up with traffic, since the throughput increase of one route cannot offset the throughput decrease of another route. Thus the rush hour is prolonged by the new link, which results in the increase of travel cost. What we can learn from the example is that link storage capacity is a key factor that has to be considered with care in the problem of network design.

Engineers and planners have to take all possible measures to avoid paradoxes in the design of transportation network. The simplest way to resolve paradoxes is to close the paradoxical links, but this may bring some criticism from the public. Besides, charging high-enough tolls on paradoxical links is equivalent to close the links. Ramp metering is a more positive strategy that can make the best use of the capacity of the new links. Time-varying tolls can totally eliminate all queues in a road network so that queue spillovers would not occur. Lane partitioning can split lanes in a link into different groups, which can allocate appropriate portions of traffic to different routes and thus maximize total throughput of the network. Adaptive traffic control can also alleviate the queue spillover problems even temporal traffic demand varies randomly.

Paradoxes may occur in both point-queue models and physical-queue models, but the mechanisms are different. In this paper, one paradox by point-queue models is caused by the interplay of different OD travelers, and the other paradox by physical-queue models is caused by the queue spillovers. It is illustrated in the second example that a serious paradox is found when the physical dimension of traffic queue is considered explicitly in the proposed model. However, there will be no paradox if the physical dimension of queue is simply ignored. It provides the justification that the spatial dimension of traffic queues should be considered explicitly in the dynamic traffic assignment models for network design problems.

In summary, the nature of generating paradox is the non-cooperative game of all travelers in choosing their departure times and routes. Paradox never occurs in the system optimal traffic assignment in either static or dynamic dimension. In fact, all measures for resolving paradox are stemmed from implementing the system optimum of traveling to some extent.

Finally, it is worthy noting that the analytical solution method presented in the paper is only appropriate for small networks. Efficient solution algorithm should be developed to solve the dynamic user equilibrium problems in large-scale road networks. Moreover, it is simplified in the model that each link is made up of a main body with constant travel time and a downstream bottleneck with a limited flow rate capacity. But in reality traffic moving speed is a function of flow density in accordance with the fundamental diagram of traffic flow. Therefore, the queuing assumptions in the paper could be further improved by the kinematic wave models (Ni et al., 2006). Another future research work might be extending the methods of modeling and managing paradoxes described in this paper, to the situations where multiple trips or tours are scheduled simultaneously in transportation networks (Heydecker and Polak, 2006; Zhang et al., 2005).

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