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AN ALGORITHM FOR MULTI-CLASS NETWORK EQUILIBRIUM PROBLEM IN PCE OF TRUCKS: APPLICATION TO THE SCAG TRAVEL DEMAND MODEL

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The analysis of mixed traffic, which includes both private and trucks has received more attention recently due to the rapid development of urban truck traffic in many cities and regions of the world. In this paper, we consider a multi-class network equilibrium model where several classes of traffic, with their own travel times, interact on the links of the network. The volume/delay functions depend on the mix of trucks and cars and other factors like the slope of the links. Hence, the resulting cost functions are nonlinear, non-smooth and asymmetric. The problem is formulated as a nonlinear, nonsmooth, variational inequality model. A linear approximation type algorithm, which uses step size based on the method of successive averages, is used to solve the problem. Numerical results are reported for a large scale problem (6 classes of traffic, 3217 zones and 99867 links).

KEYWORDS: Network equilibrium, multi-class traffic assignment, variational inequality problem

1. INTRODUCTION

The modeling of urban freight movements has received more attention recently due to the need of representing the increased truck traffic on urban road networks. The models used for transportation planning of urban areas are enhanced with the flows of trucks in the period of times studied. The traffic flow characteristics of mixed traffic of cars and trucks are quite complex since the delay depends on the proportion of the different vehicles on a given link, the grade of the link and the behavior of the drivers. The model presented in this paper is motivated by a detailed study carried out in the area covered by the jurisdiction of the Southern California Association of Governments (SCAG) by a consulting firm (Meyer, Mohaddes Associates Inc., 1999). The model that is developed in this contribution is a multi-class network equilibrium model with asymmetric costs, which is formulated as a variational inequality. It is solved by a linear approximation algorithm, which uses step sizes computed with the method of successive averages. The algorithm was implemented in a widely available transportation planning software package and was applied to the network used by SCAG for transportation planning purposes. This is a very large network consisting of 3,217 centroids, 25,430 nodes and 99,687 links.

The most difficult part of the model is that, as mentioned above the link travel times depend on the mix of trucks and cars on the link and on the slope of the link as well. The resulting link travel time functions are nonlinear and asymmetric. The volumes of trucks of different classes (that is size) are converted into passenger car equivalents, or PCE's (we will use this notation even though the reference to PCU's, passenger car units is

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common as well). The conversion factors depend on the percentage of each class of traffic on the link since the interaction between the various vehicle types is complex and quite variable.

An example of PCE conversions used may be found in the report of Meyer, Mohaddes Associates Inc. (1999) where the total PCE's on a link are a nonlinear function of three classes of trucks and three classes of private cars. The difficulty as well as the novelty of this problem arises from the fact that such a network equilibrium model was not previously studied in the literature. A new mathematical formulation as well as a solution algorithm must be developed for this model.

In this paper we focus on the model formulation, the solution algorithm and the presentation of the computational results obtained with the SCAG network. The only theoretical contributions in the literature that are relevant to the algorithm used in this model are asymmetric cost single class models. Marcotte and Zhu (1996) and Magnanti and Perakis (1994, 1997) use an LP based operator and a projection operator respectively and prove the convergence of these algorithms under certain conditions. We extend these types of algorithms for this multi-class of network equilibrium problem. The theoretical study of the convergence of the algorithm that is presented in the following is beyond the scope of this paper.

This paper is organized as follows. In Section 2, the mathematical model for the problem is defined. In Section 3, a solution method is developed for the mathematical model, while Section 4 is devoted into the application of the method in a real network. Section 5 concludes the paper.

2. MODEL FORMULATION

In this section, the notation used is introduced in order to state the mathematical formulation for the problem.

2.1 Notations

In this paper, the following notations are used. The links of the road network are designated by $a \in A$, where A is the set of links. The demand for travel by user class $m \in M$ for the origin-destination pair (i, j) is denoted as T_{ij}^m where M is the set of classes. This demands may use paths $R_{ij}^m \in R$ where R is the set of all routes $R = \bigcup_{(i,j),m} R_{ij}^m$. Table 1 summarizes the definition of sets and indices used in this paper.

TABLE 1: Sets and indices used

A	: link set of base network in period *t*	a	: link index of base network
W	: total OD pairs	w	: OD pair index
R_{ij}^m	: set of routes for pair (*i*,*j*), class *m*	r	: path index
M	: class set	m	: class index

Given the sets and indices above, the following variables and given data are defined:
The variables include:

h_r^m : The path flow of class m on the route r,

f_a^m : The total link flow of class *m* on link *a*,

C_r^m : The path travel time of class *m* on route *r*,

v_a : The total link flow in PCE (passenger car equivalence) that include flows of all classes on link *a* (which will be discussed in next section),

v_a^m : Link flow in PCE of class *m* on link *a*, $\left(v_a = \sum_m v_a^m\right)$, and

$c_a(v_a)$: The travel time on link *a* for total link flow v_a in PCE.

The given data consists of:

T_{ij}^m : The travel demand from origin *i* to destination *j* of class *m*.

2.2 Total link flow in PCE

The total link flow in PCE, that is, $v_a = f_a\left(f_a^m \mid \forall m \in M\right)$ can expressed as a very general function of link flows (in vehicles) of all classes $f_a^m, \forall m \in M$, and the parameters associated with link *a*. It is not just simply be a linear combination of the link flows of all classes such as $v_a = \sum_m \beta^m f_a^m$ where β^m is a conversion factor into PCE from the total link flow of class *m*. In particular, the following nonlinear function is considered:

$$v_a^m = f\left(f_a^m, p_a^m, vc_a, k_a, adj_a^m, l_a, g_a \mid \forall m \in M\right), \tag{1}$$

where

p_a^m is the percentage of the link flow of class *m* on link *a*,

$$p_a^m = f_a^m \Big/ \sum_{m' \in M} f_a^{m'}, \tag{2}$$

vc_a is the link flow in PCE, v_a , over the capacity of link *a*, k_a , given as,

$$vc_a = v_a / k_a, \tag{3}$$

k_a is the link capacity in PCE on link *a*,

$adj_a^m(vc_a)$ is an adjustment factor of class *m* on link *a* based on vc_a ,

l_a is the length on link *a*, and

g_a is the grade on link *a*.

This type of nonlinear function has been calibrated and validated with observed data and can be formulated in the form of a look-up table in practice. Consider the following example with v_a expressed as

$$v_a = \sum_{m \in M} PCE_a^m\left(p_a^m, k_a, adj_a^m(vc_a), l_a, g_a\right) \times f_a^m,$$

where $PCE_a^m\left(p_a^m, k_a, adj_a^m(vc_a), l_a, g_a\right)$ can be presented in the comprehensive look-up table (See Section 4 for an example).

2.3 Mathematical model

The feasible region Ω of the problem is defined as follows:

$$\sum_r h_r^m = T_{ij}^m, \quad \forall (ij), m \in M \ , \tag{4}$$

$$h_r^m \geq 0, \quad \forall m \in M, r \in R_{ij}^m \ , \tag{5}$$

where

$$f_a^m = \sum_{r \in R} h_r^m \delta_{ar}, \quad \forall a \in A, m \in M \ , \tag{6}$$

$$C_r^m = \sum_{a \in A} c_a^m (v_a) \delta_{ar}, \quad \forall m \in M, r \in R_{ij}^m \ , \tag{7}$$

where (4) are the equations of conservation of flow, (5) is the nonnegativity of the path flows, δ_{ar} is 1 if link *a* is on route *r* and is zero otherwise and (7) is the definition of path travel time of class *m* on route *r*, C_r^m.

The multi-class network equilibrium problem can be formulated as a variational inequality problem. Find $h^* \in \Omega$ such that

$$\sum_{w \in W} \sum_{m \in M} \sum_{r \in R_w} C_r^m \left(h^*\right)\left(h_r - h_r^*\right) \geq 0, \quad \forall h \in \Omega \tag{8}$$

It is well known that while the total link flow *v* in PCE is unique if $\left(c_a^m(v)\right)$ is strictly monotone, the composed volumes $\left(f^m, m \in M\right)$ may not be unique with $c_a^m(v) = c_a(v)$ for all *m*. However the strict monotonicity conditions are difficult to verify for this model. It is clear that the solution of the problem satisfies the following equilibrium conditions:

$$C_r^m \begin{cases} = u_w^m & \text{if } h_r^m > 0 \\ \geq u_w^m & \text{if } h_r^m = 0 \end{cases}, \quad \forall m \in M, r \in R_w \ , \tag{9}$$

where $u_w^m = \min_{r \in R_w} \left\{C_r^m\right\}$ is the minimum travel time for pair *w* of class *m*, which are the well known Wardrop (1952) user's optimal conditions. The derivation of the variational inequality formulation (8) from Wardrop's user optimal principle (9) is well known and may be referenced in the seminal work of Smith (1979). We may reference also the survey chapter of Florian and Hearn (1995).

3. SOLUTION ALGORITHM

In the literature, there are many recursive averaging schemes. Some were studied by Marcotte and Zhu (1996) and Magnanti and Perakis (1997). These can be used to solve the variational inequality problem with only a single class and a mapping of either LP-based operator or projection operator. The evaluation of the projection operator is equivalent to a minimization of a quadratic convex optimization problem, while the LP-based (linear approximation) operator can be used for multi-classes of demand, which can be solved by solving a shortest path problem with a network loading (a decomposable linear program problem by class) instead of the quadratic optimization problem. We choose the LP-based operator for the development of the solution algorithm, which is given as follows.

3.1 Multi-class mixed flow network assignment algorithm

Step 0 : Initialization. Start with $l = 0,\ f_a^{m,l} = 0$.

Step 1: Compute percentage of link flow, v/c ratio and link flow in PCE.

$$p_a^{m,l} = f_a^{m,l} \Big/ \sum_{m' \in M} f_a^{m',l}\ , \quad \forall a \in A, m \in M\ ,$$

$$vc_a^l = vc_a^{l+1} \big/ k_a\ , \quad \forall a \in A\ ,$$

$$v_a^{m,l} = PCE_a^m \left(p_a^{m,l}, k_a, adj_a^m (vc_a^l), l_a, g_a \right) \times f_a^{m,l}\ .$$

Step 2: Computation of link cost. $c_a(v_a), \forall a \in A$.

Step 3: Computation of the shortest path flow problem for each class *m*.

$$\min \sum_{w \in W} \sum_{r \in R_w^m} C_r^{l,m} \bar{h}_r^m\ ,$$

$$\text{s.t.} \ \sum_{r \in R_w^m} \bar{h}_r^m = T_w^m\ , \quad \forall w \in W, m \in M\ ,$$

$$\bar{h}_r^m \geq 0, \quad \forall m \in M, r \in R_w^m, w \in W\ .$$

Step 4: Computation of link flow $\bar{f}_a^m = \sum_{r \in R} \bar{h}_r^m \delta_{ar}\ , \quad \forall a \in A, m \in M$.

Step 5: Computation of step size (MSA)

$$f_a^{m,l+1} = f_a^{m,l} + \left(\bar{f}_a^{m,l} - f_a^{m,l} \right) \Big/ (l+1)\ , \quad \forall a \in A, m \in M\ .$$

Step 6: Apply a stopping criterion.

Step 7: $l = l + 1$. Go to Step 1.

4. APPLICATION

This algorithm was applied to solve the particular multi-class network equilibrium problem in PCE in the network used by SCAG for transportation planning, which was mentioned in Section 1. The algorithm was implemented in the EMME/2 (INRO, 1998) transportation planning software by using its macro language and the judicious use of the multi-class assignment module. Step 2, Step 3 and Step 4 of the algorithm use the first iteration of a symmetric cost multi-class assignment module to compute shortest paths and load the demand for different classes, while Step 1 and Step 5 are implemented with network calculations. Step 6 is coded directly with a macro language procedure.

There are three classes of passenger vehicles and three classes of trucks. The well known BPR volume delay function was used in its original forms:

$$c_a(v_a) = \beta_a \left(1 + 0.15 \times (v_a / k_a)^4 \right),$$

where β_a and k_a (practical capacity) are both parameters. The six classes of demand of traffic are listed in Table 2.

TABLE 2: Six classes of demands

Class ID	Descriptions
1	Passenger cars of one person demand that can not access HOV links
2	Passenger car of two person demand that can access whole network
3	Passenger cars of three person or more demand that can access whole network
4	Light-heavy duty trucks that can not access HOV links
5	Medium-heavy duty trucks that can not access HOV links
6	Heavy-heavy duty trucks that can not access HOV links

The trip demands for the six classes of traffic considered in the model are as follows:

Class 1 :	5,119,365	Passenger cars
Class 2 :	2,070,009	Passenger cars
Class 3 :	384,592	Passenger cars
Class 4 :	79,583	Light-Heavy trucks
Class 5 :	51,794	Medium-Heavy trucks
Class 6 :	34,809	Heavy-Heavy trucks

This is probably one of the largest multi-class assignment problems used in the practice of transportation planning. The three classes of trucks are classified as

1. Light-Heavy : 8,500 to 14,000 GVW
2. Medium-Heavy : 14,000 to 30,000 GVW
3. Heavy-Heavy : over 30,000 GVW

In the computation, the various variables and data information for each link are stored, as shown in Table 3.

TABLE 3: Partial link attributes

Notation	Description
v_a	Current total equivalent car flow
f_a^m	Successive average flow on link *a* of class *m*
g_a	Attribute as the grade for the link

Thus total link flow in PCE v_a on link *a* is a nonlinear function of three heavy duty trucks and three auto vehicles which is defined as:

$$PCE_a^m\left(p_a^m, k_a, adj_a^m(vc_a), l_a, g_a\right) = PCE_a^m\left(p_a^m, l_a, g_a\right) \times adj_a^m\left(vc_a, p_a^m\right),$$

where $adj_a^m\left(vc_a, p_a^m\right)$ and $PCE_a^m\left(p_a^m, l_a, g_a\right)$ $\left(PCE_a^m\left(p_a^m, l_a, g_a\right) = 1, \text{ for } m = 1,2,3\right)$ are defined in Table 4 and Table 5. Thus the nonlinear functions of (1) are defined in the form of look-up tables instead of the continuous functions. These functions have been validated with observed data, as reported in Meyer, Mohaddes Associates Inc. (1999).

TABLE 4: adj_a^m values *m* = 4, 5, 6

vc_a	p_a^4			p_a^5			p_a^6		
	0-5	5-10	>10	0-5	5-10	>10	0-5	5-10	>10
<0.5	0.60	0.66	0.90	0.66	0.77	0.93	0.90	0.77	0.93
0.5-1.0	0.77	0.89	1.15	0.89	1.01	1.20	1.15	1.01	1.20
1.0-1.5	1.10	1.20	1.30	1.20	1.25	1.34	1.30	1.25	1.34
1.5-2.0	1.00	1.05	1.22	1.05	1.22	1.25	1.22	1.22	1.25
>2.0	1.19	0.66	1.26	1.05	1.24	1.29	1.26	1.24	1.29

Two measures of the convergence of the algorithm are provided. The first measure (M1) is the relative difference d^l at iteration *l* between the volume at iteration *l* and successive average volume at iteration *l*:

$$d^l = \sum_a \sum_m \left| \bar{f}_a^{m,l} - f_a^{m,l} \right| \Big/ \sum_a \sum_m \bar{f}_a^{m,l}$$

TABLE 5: Heavy duty truck PCE values $PCE_a^m\left(p_a^m, l_a, g_a\right)$

p_a^m	l_a	$m = 4$			$m = 5$			$m = 6$		
		g_a			g_a			g_a		
		0-2	3-4	>4	0-2	3-4	>4	0-2	3-4	>4
0-5	≤1	2.0	4.2	6.4	3.4	6.9	8.8	4.3	8.0	11.3
	>1	2.0	5.5	7.5	5.2	8.4	10.7	6.7	10.5	13.5
5-10	≤1	2.0	3.4	4.8	3.1	5.0	6.4	3.5	5.8	8.8
	>1	2.0	4.2	5.3	3.9	5.9	7.8	4.8	7.8	13.5
>10	≤1	2.0	3.3	4.1	2.8	4.5	6.0	3.2	5.1	8.3
	>1	2.0	3.5	5.0	3.7	5.3	7.5	4.0	7.5	12.5

The second measure (M2) is the relative gap $rgap^l$ at iteration l computed with the flow $\bar{f}_a^{m,l}$ which is the "all-or-nothing" assignment on shortest paths and the last flow $\bar{f}_a^{m,l-1}$ weighted by the current travel time $c_a\left(\bar{v}_a^l\right)$:

$$rgap^l = \sum_a \sum_m c_a\left(\bar{v}_a^l\right)\left|\bar{f}_a^{m,l-1} - \bar{f}_a^{m,l}\right| \Big/ \sum_a \sum_m c_a\left(\bar{v}_a^l\right)\bar{f}_a^{m,l-1}$$

It is well known that if $d^l \to 0$ or $rgap^l \to 0$ as $l \to \infty$, then one can obtain an equilibrium solution. This solution procedure was implemented in the EMME/2 software package as a macro. The algorithm converges on this network, as shown in Figure 1, where M1 and M2 are the two convergence measures mentioned above.

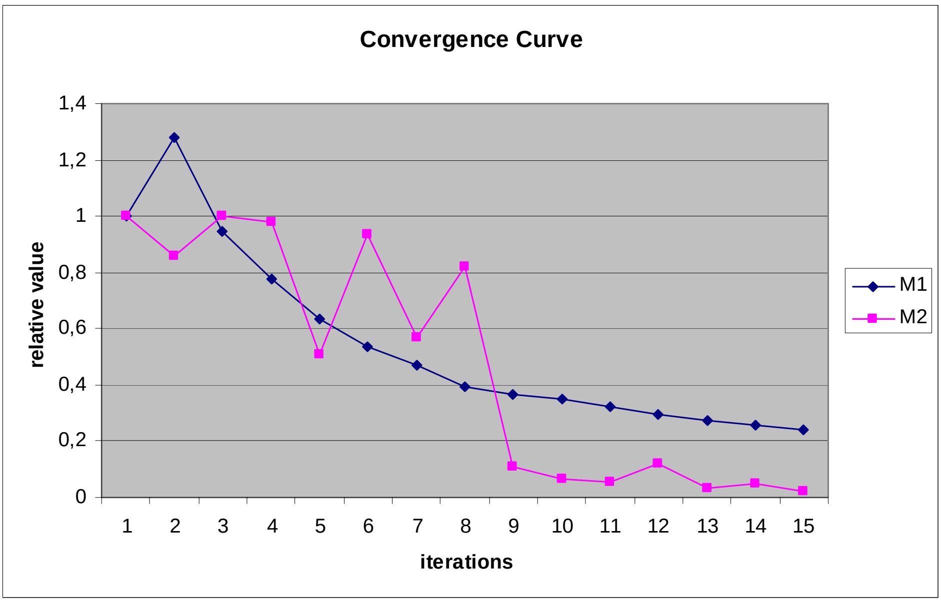

FIGURE 1: Convergence of the solution procedure

The computations require about 40 min per iteration on a SUN SPARC ULTRA 5 workstation and about 7 minutes on an IBM Thinkpad T40 based on an Intel Centrino

1.6 Mhz. If the algorithm is used in a travel demand forecasting model, one may need only 4-6 iterations for a reasonable convergence of an inner loop. The assignment results are shown in Figures 2 and 3 where private vehicles (that is, m = 1, 2, 3) are not shown.

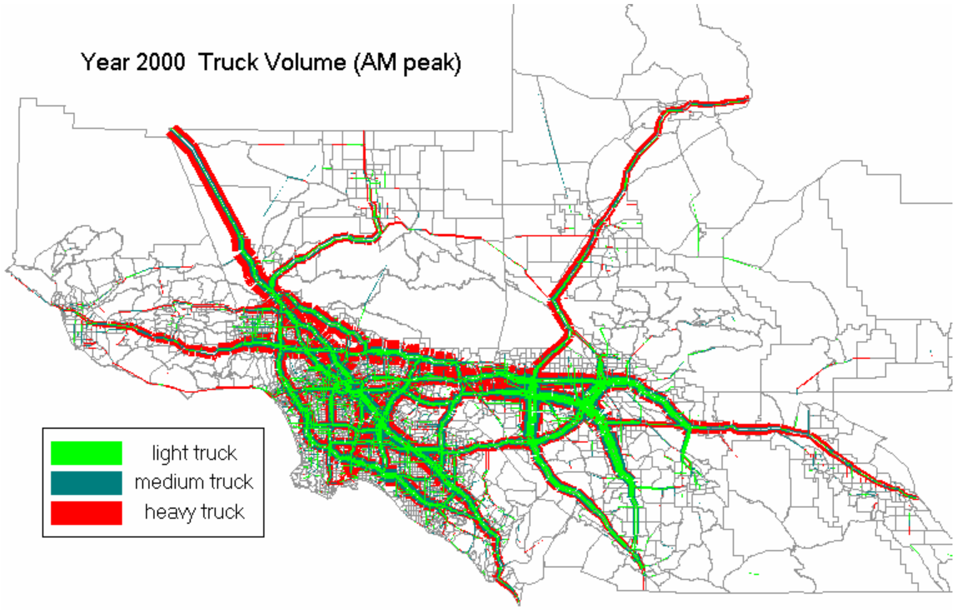

FIGURE 2: Three classes of trucks – general view

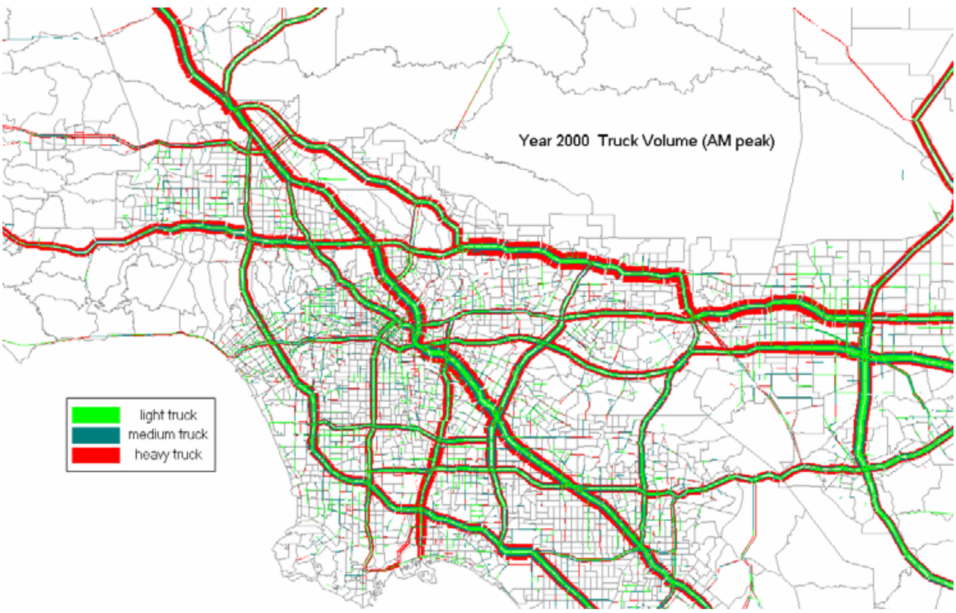

FIGURE 3: Three classes of trucks – a window

5. CONCLUSION

In this paper, the issue of a multi-class mixed car and truck traffic network equilibrium problem was considered. Based on the analysis of the problem, a variational inequality problem was formulated and an adapted successive averaging method was developed for solving a six-class variational inequality problem. Then the linear approximation based method was applied to a real network with good numerical results. This model is actually used in practice.

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