Transportmetrica, Vol. 1, No. 2 (2005), 151-159

FINITE ELEMENT MODELING AND OPTIMIZATION OF TRAFFIC FLOW NETWORKS

G.A. MOHR[1]

Received 13 July 2004; received in revised form 12 November 2004; accepted 12 December 2004

The classical linear rule for traffic flow is used to form simple two freedom elements of traffic flow networks. A simple network is used to demonstrate modeling of flow variations caused by modification of the network. The steepest descent method is used to seek an optimum network and the results compared to those of Mohr's Flow Ratio Design (FRD) method. Similar results are obtained when a simple bilinear flow rule is used for flow in the network.

KEYWORDS: Traffic flow, finite elements, optimization, steepest descent, flow ratio design, bilinear flow rule

1. INTRODUCTION

The simplest possible finite elements have two nodes and freedoms and are linear. These can, for example, be used to model DC networks (Jennings, 1977; Zienkiewicz 1977) and Mohr and Milner (1987) give a very short PC program for this purpose.

There are a few network problems in the field of Operations Research (Budnick et al., 1977; Hillier and Lieberman, 1980). Perhaps the most widely studied is the distribution problem and Mohr (2000b) uses two freedom linear elements to model this, then optimizing these models using the steepest descent method.

In the present work the classical linear rule for traffic flow (Salter, 1976) is used to form two freedom linear elements of traffic flow networks. Then the effects of modifications of a network are modeled and the steepest descent method (Mohr, 1992, 2000b) is used to obtain a simpler optimal network with reduced total travel times and the results are verified using linear programming.

Then the Flow Ratio Design (FRD) method (Mohr, 2004) is demonstrated on a distribution problem and then applied to the simple traffic flow network example of the present paper. FRD is an optimality criterion method based on the Fully Stressed Design (FSD) method, a simple and widely used method of improving finite element models of structures in which the solution is iterated, element stresses being calculated at each iteration and used to adjust the element dimensions according to the ratio of the maximum stress in the element and the stress limit for its constitutive material. In many problems convergence to an approximately optimal solution is obtained after a few iterations (Gallagher and Zienkiewicz, 1973; Mohr, 1999b, 2000a).

Much motivation for FRD as an optimality criterion is given by the classical work of Michell (1904) in which it is shown that minimum weight planar truss structures will have constant strain magnitude. Using an argument based on generalized constraints with slack variables Mohr (1992) also obtains Michell's constant strain condition for optimality, also including a term to allow for vanishing members. The flow ratio method of the present work is, in part, based on this work, except that a 'median' flow value is used in the ratio calculation.

[1] International Arts and Sciences College, 68 Tulip Grove, Cheltenham VIC 3192, Australia. Corresponding author (Tel: +613-95848515).

2. LINEAR TRAFFIC FLOW ELEMENTS

The classical linear flow rule for traffic flow is

$$v_{ij} = V_{ij}\left(1 - k_{ij} / K_{ij}\right), \quad (1)$$

where v_{ij} and k_{ij} are the element traffic velocity and density and V_{ij} and K_{ij} are respectively the element free flow velocity and jam density. Then the equations for each element or link are

$$\begin{Bmatrix} q_i \\ q_j \end{Bmatrix} = \left(K_{ij}V_{ij} / L_{ij}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_i \\ P_j \end{Bmatrix}, \quad (2)$$

where q_{ij} is the link flow, $q_i = -q_{ij}$ and $q_j = q_{ij}$ are the flows at each end, L_{ij} is the element length and P_i, P_j are arbitrary potentials at the element nodes and these have the dimension of length. Thus the problem is analogous to that of a DC network, where now $L_{ij} /(K_{ij}V_{ij})$ plays the role of the element 'resistance', and to model a network the element matrices are deployed according to their node numbers to form a system matrix for the network. Therefore elements can have flow in only one direction but either 'forwards' (to node *j*) or in reverse (to node *i*).

Figure 1 shows a simple example network with the nodal flows shown, positive flows being inwards. The same free flow velocity $V_f = 60$ is assumed for all elements. A single boundary condition $P = 0$ is set at node 8 as a datum and the in and out flows are the 'load' data. The system equations are solved to determine the nodal potentials P_i from which the element flows are calculated using

$$q_{ij} = R_{ij}\left(P_i - P_j\right), \text{ where } R_{ij} = K_{ij}V_f / L_{ij}. \quad (3)$$

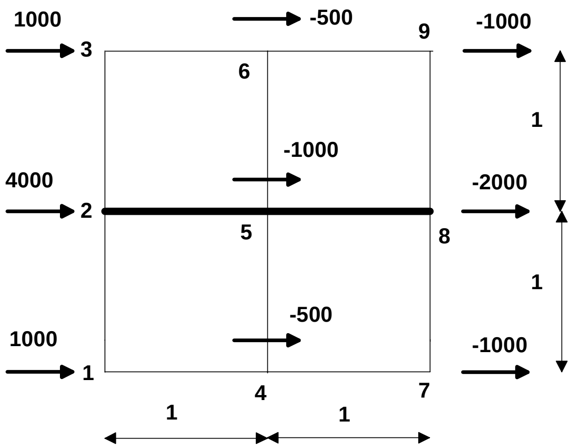

FIGURE 1: Simple traffic flow network. (a) $V_f = 60$, $K_{ij} = 100$ for all links, except that K_{25} and K_{58} = 300, (b) K_{25} and K_{58} are increased to 600

Then solving the quadratic equation

$$q_{ij} = k_{ij}v_{ij} = k_{ij}V_{\mathrm{f}}\left(1 - k_{ij} / K_{ij}\right), \tag{4}$$

two roots k_a and k_b and their corresponding velocities v_a and v_b are obtained. In the present work the feasible root is the larger velocity v_b and using this the total travel time (per unit time) in the network is calculated as

$$T = \sum \left|q_{ij}\right| L_{ij} / \left|v_b\right| . \tag{5}$$

The results are shown in Table 1. In case (b) we have upgraded the centre links 2-5, 5-8 and the flow changes are as expected, the flows in these centre links increasing and the total travel time in the system decreasing.

TABLE 1: Solutions for the link flows in Figure 1

	(a) K_{25}, K_{58} = 300			(b) K_{25}, K_{58} = 600		
Link	Flow	k_b	v_b	Flow	k_b	v_b
1-2	-178.6	3.07	-58.16	60.4	1.02	59.39
2-3	178.6	3.07	58.16	-60.4	1.02	-59.39
1-4	1178.6	26.86	43.89	939.6	19.44	48.34
4-5	-142.9	2.44	-58.54	-192.3	3.32	-58.01
3-6	1178.6	26.86	43.89	939.6	19.44	48.34
5-6	142.9	2.44	58.54	192.3	3.32	58.01
2-5	3642.9	84.54	43.09	4120.9	79.11	52.09
5-8	2357.1	46.49	50.70	2736.3	49.73	55.03
4-7	821.5	16.37	50.18	631.9	11.96	52.82
6-9	821.5	16.37	50.18	631.9	11.96	52.82
7-8	-178.6	3.07	-58.16	-368.1	6.57	-56.06
8-9	178.6	3.07	58.16	368.1	6.57	56.06
Total travel time	234.63 hours			213.44 hours		

To three decimal places the solutions for the nodal potentials in case (a) were, in order

0.304, 0.333, 0.304, 0.107, 0.131, 0.107, -0.030, 0, -0.030

so that the flow in element 25 is approximately = 0.202 × 300 × 60 / 1 = 3636.

3. OPTIMIZING THE NETWORK

To optimize such FEM models the method of steepest descent (Mohr, 2000b) is used, first defining an 'element access' parameter A_{ij} and at the beginning of each search $A_{ij}^0 = K_{ij}V_{\mathrm{f}} / L_{ij}$, again assuming the free flow velocity to be the same for all elements. The merit function is the total travel time of equation (5). Using perturbations of each A_{ij}^0 in turn the gradient vector for search is given by

$$\{g\} = \left\{\delta T / \delta A_{ij}\right\} \text{ where } \delta A_{ij} = (M - 1)A_{ij}^0 , \tag{6}$$

with M a move limit for which the value 1.1 is used in the present work.

Using this to search in the direction of steepest descent we write

$$\left\{A_{ij}^*\right\} = \left\{A_{ij}^0\right\} - S\{g\}, \tag{7}$$

using trial values of the step length S to locate a turning point in T. If, during search, the magnitude of a link flow is less than a small value $q_{\min}$ then we set $A_{ij} = 0$ for that link and omit it from the model, also setting a flag which returns to calculation of a new gradient vector once the other link flows and thence T have been calculated. Note that such element omission is disallowed during calculation of the gradient vector.

For the example of case (b) of Figure 1, using $q_{\min}$ = 10 and successive searches S = 0.9, 200 and 0.00001 (the small last search gives no reduction in T so search is terminated but some figures 'round off'), the resulting solution has only eight links and is given in Table 2.

TABLE 2: Steepest descent solution for optimum network in Figure 1

Link	q_{ij}	K_{ij}	k_b	v_b
1-4	1000	100	21.132	47.321
3-6	1000	100	21.132	47.321
2-5	4000	600	76.393	52.36
5-8	3000	600	55.051	54.495
4-7	500	100	9.175	54.495
6-9	500	100	9.175	54.495
7-8	-500	100	9.175	-54.495
8-9	500	100	9.175	54.495

The initial value of the total flow was T = 213.44, and the final value is T = 210.41, a significant saving considering that four elements/links have been omitted.

This result was verified using linear programming with simple 'split' constraints for the equality constraints for each node. For node 1, for example, these are

$$-q_{12} + q_{14} + s_1 = 1000 \text{ and } q_{12} - q_{14} + s_1 = -1000\,, \tag{8}$$

where s_1 is the first slack variable. The constraints for one node are omitted, as is customary for the distribution problem, because of the implicit constraint total inflows = total outflows.

4. FLOW RATIO DESIGN METHOD

As an introduction to this we begin with the familiar distribution problem for which element matrices are also given by equation (2) if the constitutive parameter $K_{ij}V_{ij} / L_{ij}$ is replaced by $1/c_{ij}$ where c_{ij} is the unit distribution cost for route *ij* (Mohr, 1999a). The network can then be optimized using the same steepest descent method used in the present work, now omitting routes in which $q_{ij} < 0$ occurs during search.

Alternatively, the simple Flow Ratio Design (FRD) method (Mohr, 2004) can be used to obtain approximate optima. In the distribution problem the element unit costs are adjusted at each iteration using

$$c_{ij}^* = c_{ij}\left(q_{\text{m}} / \left|q_{ij}\right|\right) \text{ where } q_{\text{m}} \cong q_{\text{av}} / 2\,,\; q_{\text{av}} = Q / N \tag{9}$$

where Q is the total flow in the network of N routes and q_{m} is the 'median' flow. Then to obtain the minimum solution lower and upper cost limits

$$c_{\text{L}} = c_{\text{av}} / 40 - c_{\text{av}} / 10\,,\; c_{\text{U}} = 10^6 \text{ where } c_{\text{av}} = \sum c_{ij} / N \tag{10}$$

are used.

Observing these limits iteration proceeds and some routes are eliminated when their route flows q_{ij} are less than $q_{\min} = 0.001$, prior to calculating the total distribution cost

$$C = \sum \left|q_{ij}\right| c_{ij}^0\,, \tag{11}$$

where c_{ij}^0 are the initial unit costs for each route.

Table 3 shows the route flows obtained using 16 FRD iterations with $q_{\mathrm{m}} = 10$ for a 3×4 problem with supply flows (i = 1,2,3) of 60, 80, 60 and demand flows (j = 4,5,6,7) of -50, -40, -70, -40. Here for minimization $c_{\mathrm{L}} = 0.2 \cong c_{\mathrm{av}}/15$ was used and the minimum solution (row 4) is an upper bound, but close to the exact minimum (row 3) obtained by linear programming.

TABLE 3: FRD solution for a distribution problem

Route	1-4	1-5	1-6	1-7	2-4	2-5	2-6	2-7	3-4	3-5	3-6	3-7	*C*
c_{ij}	2	5	4	5	1	2	1	4	3	1	5	2	
MIN	50	0	0	10	0	10	70	0	0	30	0	30	330
FRD	40	0	0	20	10	0	70	0	0	40	0	20	340

Note in passing that for the distribution problem dual models for which c_{ij} is simply replaced by its reciprocal to form the element equations also exist (Mohr, 1999a, 2000b). Indeed it is found that, using FRD, maximum solutions for the primal distribution problem and minimum solutions for the dual problem converge towards the same result, an intermediate saddle point solution (Mohr, 2004).

5. APPLICATION OF FRD TO TRAFFIC FLOW MODELS

For traffic flow networks the constitutive parameter is $R_{ij}=K_{ij}V_{ij}/L_{ij}$. Only K_{ij} is variable in the current problem (of Figure 1) and this is updated using

$$K_{ij}^{*} = K_{ij}\left(\left|q_{ij}\right| / q_{\mathrm{m}}\right). \tag{12}$$

For the problem of Figure 1 (case (b)) the total flow is 6000 with 12 links, giving q_{av} = 500 and this value is used for q_{m}.

Then using lower and upper limits $(K_{ij})_{\mathrm{L}}$ = 100, $(K_{ij})_{\mathrm{U}}$ = 10000 and omitting links with $|q_{ij}| < q_{\min}$ = 100, the solution of Table 4 is obtained for the link flows after eight iterations.

TABLE 4: FRD solution for the optimum network in Figure 1

Link	q_{ij}	K_{ij}	k_b	v_b
1-4	1000	10000	16.695	59.900
3-6	1000	10000	16.695	59.900
2-5	4000	10000	67.117	59.597
5-8	3000	10000	50.253	59.698
4-7	500	143.72	8.882	56.292
6-9	500	143.72	8.882	56.292
7-8	-500	100	9.175	-54.495
8-9	500	100	9.175	54.495

The link/element flows are the same as in Table 2, but the densities and velocities differ, and thence the total travel time T = 186.87 is less than obtained using steepest descent because FRD allows the element jam densities to vary (as the only variable part of the element parameter R_{ij}). The values thus are close to or equal to the limiting values, an interesting but somewhat impractical solution. Thus to obtain the results of Table 2 for the element densities and velocities they must be calculated using the original values of the link parameters R_{ij}.

6. APPLICATION OF THE BILINEAR FLOW RULE

Next the bilinear traffic flow rule shown in Figure 2 (Mohr, 1983) is applied to the simple traffic network analysis of Figure 1. In this there is a plateau in the traffic flow rate at velocity V_p and here V_p = 55 is assumed for all links. For rural traffic the flow-density relationship is a little more complicated than that for the classical linear rule but for the case of urban traffic equation (4) still applies and is again used in the present work.

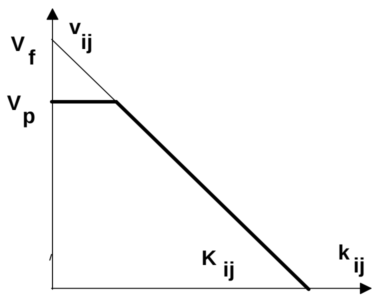

FIGURE 2: Bilinear flow rule

In programming this merely requires that any links for which v_b is found to exceed V_p have v_b set equal to sgn(v_b)V_p and the corresponding density is calculated as k_b = sgn(v_b)q_{ij}/V_p.

The results are shown in Table 5 where, as expected, they are similar to those of Table 1 except that six links in case (a) have been restricted to velocity V_p whilst in case (b) seven links have this limiting velocity. In both cases total travel time T in the network has increased slightly, as expected.

TABLE 5: Solutions for the link flows in Figure 1

	(a) K_{25}, K_{58} = 300			(b) K_{25}, K_{58} = 600		
Link	Flow	k_b	v_b	Flow	k_b	v_b
1-2	-178.6	3.25	-55	60.4	1.10	55
2-3	178.6	3.25	55	-60.4	1.10	-55
1-4	1178.6	26.86	43.89	939.6	19.44	48.34
4-5	-142.9	2.60	-55	-192.3	3.50	-55
3-6	1178.6	26.86	43.89	939.6	19.44	48.34
5-6	142.9	2.60	55	192.3	3.50	55
2-5	3642.9	84.54	43.09	4120.9	79.11	52.09
5-8	2357.1	46.49	50.70	2736.3	49.75	55
4-7	821.5	16.37	50.18	631.9	11.96	52.82
6-9	821.5	16.37	50.18	631.9	11.96	52.82
7-8	-178.6	3.25	-55	-368.1	6.69	-55
8-9	178.6	3.25	55	368.1	6.69	-55
Total travel time	235.66 hours			214.24 hours		

Next steepest descent is applied to case (b) to optimize the network. Again using $q_{\min}$ = 10 for all links the first search terminates with search length S = 0.9 when two links are eliminated and T = 210.65. The second search yields a minimum value T = 210.27 when S = 34 and when S = 44 two more links are eliminated, after the usual last ‘unproductive’ search of S = 0.00001 giving the solution shown in Table 2 and the numerical model now behaves like that of a statically determinate structure and no change occurs with further search.

Note that the final solution with eight links has T = 210.41 whereas the intermediate solution with ten links had T = 210.27. In this intermediate solution the element parameters A_{ij} have been altered by equation (7). As all links have L_{ij} = 1 and V_f = 60 the corresponding values of the element jam densities are $K_{ij} = A_{ij}^{*}/V_f$, giving the solution shown in Table 6. For practical purposes, however, the simpler solution of Table 2 is probably preferable as the two further links eliminated must result in greater cost savings than the marginal reduction in total travel time given by the solution of Table 6.

TABLE 6: Steepest descent solution for optimum network in Figure 1

Link	q_{ij}	K_{ij}	k_b	v_b
1-4	1000	100	21.132	47.321
4-5	-42.2	94.3	0.766	-55
3-6	1000	100	21.132	47.321
5-6	42.2	94.3	0.766	55
2-5	4000	600	76.393	52.361
5-8	2916	796	53.336	54.666
4-7	542	74.6	10.045	53.973
6-9	542	74.6	10.045	53.973
7-8	-458	113	8.325	-55
8-9	458	113	8.325	55

Note that at the start of the second search leading to the solution of Table 6 the solution is similar to that of Table 6, except that the two small flows in links 4-5 and 5-6 have magnitude 150, link 5-8 has flow 2700, links 4-7 and 6-9 flows of 650 and links 7-8 and 8-9 flows of magnitude 350. The flow densities alter correspondingly but the flow velocities are still very similar to those of Table 6 and the total travel time is T = 210.65, slightly more than that of the ‘statically determinate’ eight link solution of Table 2. Note too that this solution has the original jam densities for case (b) of Figure 1.

Next V_p = 50 is assumed for all links, giving a much increased total travel time T = 226.13 for the network of Figure 1. Applying steepest descent two links are eliminated with search length S = 0.8, giving the same ten link solution just described, that is with q_{45} = 150 etc., except that all links but 1-4 and 3-6 have the limiting velocity magnitude V_p and T = 222.27.

This solution is obtained by following the standard procedure of commencing the next search with S = 0.00001 to see if reduction in the objective function T is possible. In this case no change in the solution is obtained with further search.

The eight link solution of Table 2 is obtained with the move limit of equation (6) M = 1.2 and searches with search lengths of S = 1.0 and S = 0.6, except that six links have velocity V_p with corresponding density and now the total travel time is T = 222.27, as for the ten link solution with V_p = 50.

Finally the FRD procedure is applied with V_p = 55 when, after seven iterations, a similar result to that of Table 4 is obtained, except that all but the last two links have v_b = 55 with corresponding changes in k_b. If the original element parameters, not those altered during iteration by equation (7), are used in calculating k_b and v_b the solution is identical to that of Table 2, again yielding T = 210.41.

With V_p = 50 FRD gives the same flows as in Table 4 but all the velocities have magnitude V_p, with corresponding changes in the values of k_b. Using the original element parameters, the velocities in the first two links are reduced to 47.32 and T = 222.27.

7. CONCLUSIONS

1. Using linear elements with two freedoms based on the classical linear traffic flow rule simple finite element models of traffic flow networks are obtained. These readily show the effects of variations in the parameters for some elements of a network.
2. A simple steepest descent procedure can be used to search for an optimum network which minimizes the total travel time, some elements of the network being omitted during the successive searches. In the simple example considered here the solution is identical to that obtained by linear programming.
3. An iterative optimality criterion method, the Flow Ratio Design (FRD) method, can also be used to seek optima. In this the element parameter R_{ij} is updated at each iteration according to the ratio of its flow and an average flow value for the network. The FRD method is first demonstrated for a distribution problem, showing that good approximations to the optimum solution can be obtained. The method is then applied to the simple traffic flow network example, yielding the same link flows and also the same link flow densities and velocities provided that the original element parameters are used in the calculation.
4. Both procedures work effectively when a bilinear flow rule is used but when $V_p = 50$ steepest descent eliminates only two links unless a larger move limit $M = 1.2$ is used in calculating the gradient vector for search.
5. The FRD method, corresponding as it does to Fully Stressed Design in structural mechanics, should be applicable to a wide range of flow problems in both operations research and computational fluid mechanics.
6. The major contributions of the present paper, therefore, are
   (a) The use of simple linear elements with both linear and bilinear flow rules. It should be noted that finite elements have also been applied to the analysis of traffic flows by Yang et al. (1994), Wong et al. (1998) and Wong et al. (2003).
   (b) The use of a steepest descent procedure to optimize these models.
   (c) The use of the Flow Ratio Design method to obtain approximate optima.
   These procedures have been tested on larger problems (Mohr, 2000b, 2004).

REFERENCES

Budnick, F.S., Mojena, R., Vollmann, T.E. and Irwin, R.D. (1977) *Principles of Operations Research for Management*. Irwin, Homewood, IL.

Gallagher, R.H. and Zienkiewicz, O.C. (1973) *Optimum Structural Design*. Wiley, London.

Hillier, F.S. and Lieberman, G.J. (1980) *Introduction to Operations Research*. 3rd Edition, Holden-Day, Oakland, CA.

Jennings, A. (1977) *Matrix Computation for Engineers and Scientists*. Wiley, Chichester.

Michell, A.G.M. (1904) The limits of economy in frame structures. *Philosophical Magazine (Series 6)*, **8**, 589-597.

Mohr, G.A. (1983) A bilinear rule for macroscopic traffic flow. *Australian Road Research*, **13**, 38-40.

Mohr, G.A. (1992) *The Finite Element Method for Solids, Fluids, and Optimization*. Oxford University Press, Oxford.

Mohr, G.A. (1999a) Finite element modeling of distribution problems. *Applied Mathematics and Computation*, **105**, 69-76.

Mohr, G.A. (1999b) Finite element solutions for optimal triangular plates. *International Journal of Mechanical Sciences*, **41**, 1289-1300.

Mohr, G.A. (2000a) Finite element solutions for optimal square plates. *International Journal of Mechanical Sciences*, **42**, 2337-2345.

Mohr, G.A. (2000b) Optimization of primal and dual network models of distribution. *Computer Methods in Applied Mechanics and Engineering*, **188**, 135-144.

Mohr, G.A. (2004) Flow ratio design of primal and dual network models of distribution. *Australian and New Zealand Journal of Industrial and Applied Mathematics*, **45**, 573-583.

Mohr, G.A. and Milner, H.R. (1987) *A Microcomputer Introduction to the Finite Element Method*. Heinemann, London.

Salter, R.J. (1976) *Highway Traffic Analysis and Design*. MacMillan, London.

Wong, S.C., Lee, C.K. and Tong, C.O. (1998) Finite element solution for the continuum traffic equilibrium problems. *International Journal for Numerical Methods in Engineering*, **43**, 1253-1273.

Wong, S.C., Du, Y.C., Ho, H.W. and Sun, L.J. (2003) Simultaneous optimization formulation of a discrete/continuous transportation system. *Transportation Research Record*, **1857**, 11-20.

Yang, H., Yagar, S. and Iida, Y. (1994) Traffic assignment in a congested discrete/continuous transportation system. *Transportation Research*, **28B**, 161-174.

Zienkiewicz, O.C. (1977) *The Finite Element Method*. 3rd Edition, McGraw-Hill, London.