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EXISTENCE AND UNIQUENESS OF A SOLUTION FOR THE MULTI-CLASS USER EQUILIBRIUM PROBLEM IN A CONTINUUM TRANSPORTATION SYSTEM

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Consider a city with several facilities competing for multi-class users that are distributed continuously over space. Within the city region, the road network is relatively dense and is considered as a continuum. The demand distribution function is formulated as a logit-type function for modeling the probabilistic choice behavior of facilities for different classes of users. A minimization problem that describes this multi-class and multi-facility choice is set up for finding the user equilibrium flow pattern. This paper aims to study the existence and uniqueness of a solution of this multi-class and multi-facility user equilibrium problem with demand distribution. The assumptions and conditions for the existence and uniqueness of solution are given.

KEYWORDS: Continuum transportation system, multi-class, multi-facility, user equilibrium, existence, uniqueness

1. INTRODUCTION

In a transportation system, there are different classes of users who behave differently in both route and facility choice. These different classes of users perceive cost differently and are required to pay different charges. Therefore, it is reasonable to include this multiple user class feature in the network equilibrium model to closely reproduce the real-life situation and generate realistic and sensible forecasts of traffic flows and market shares. Dafermos (1972) was one of the first to formulate the multi-class network equilibrium problem. She considered the multi-class problem for a discrete network with links that had different cost functions for different classes of users. Recently, Vliet et al. (1986) extended the work to include more general cost functions, and Lam and Huang (1992) considered a combined distribution and assignment problem. More recently, Wong et al. (2003) studied a combined distribution, hierarchical mode choice, and assignment network model with multiple user and mode classes, and Wu et al. (2006) considered a multi-class network equilibrium model in which several classes of traffic interact on the links with nonlinear, non-smooth and asymmetric cost functions.

The multiple user class problems discussed in the previous paragraph are mainly based on the discrete approach in which a road is modeled as a discrete link in a network. An alternative but promising approach is the use of continuum modeling, in which the dense transportation network is approximated as a continuum. This approach is very useful for macroscopic modeling of a dense transportation network in a large area, in which the detailed network information is not available or the set up of a discrete network is too demanding. Ho et al. (2006) have incorporated multiple user classes into this continuum modeling approach. In their study, the distribution of demand in a continuum model with competitive facilities, which was first introduced by Wong and Sun (2001), was also

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considered. With the extension of multiple user classes in the continuum model, many real-life applications can be considered; for instance, the study of the market shares of competing airports within a multi-airport region with multi-class passengers can be conducted (Loo et al., 2005).

The existence and uniqueness of a solution are always key concerns of researchers in evaluating the applicability of the developed model. This is also true for the continuum modeling approach. Beckmann (1952) and Yang (1996) studied the existence and uniqueness of a solution for continuum user equilibrium models with a single class of users and fixed demand distribution. This paper shows the existence and uniqueness of a solution for the multiple user class model that was proposed by Ho et al. (2006).

The paper is organized as follows. The definitions and notation are introduced in Section 2. In Section 3, the model formulation and governing equations of the multi-class user equilibrium problem that was introduced by Ho et al. (2006) are presented. The proof of existence and uniqueness of a solution for this multi-class user equilibrium problem are given in Sections 4 and 5 respectively. Section 6 provides concluding remarks.

2. DEFINITIONS AND NOTATION

Figure 1 shows the modeled city region. There are several facilities, which are sufficiently compact when compared with the whole region. Different classes of users are continuously distributed over space, and they are free to choose any facility based on the cost incurred when patronizing that facility. Users, which can be passengers or freight, move to their chosen facility along an optimal route by minimizing a transportation cost that takes into consideration the effect of traffic congestion. Let the number of user classes be M, the region of the study area be Ω, and the boundary be Γ. The locations of the facilities are O_n, $n = 1, 2, \ldots, N$, where N is the number of facilities in the study area. To avoid singularity at the facilities, it is assumed that each of the facilities is of finite size and enclosed by a boundary, Γ_{nc}. The following notation is used throughout the paper.

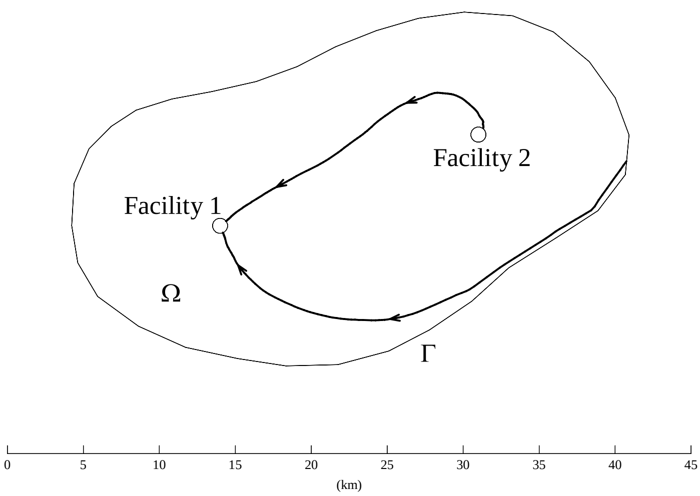

FIGURE 1: The modeled city region of this study

$\mathbf{f}_{mn}(x,y)$ = the flow vector at $(x,y)\in\Omega$ for class *m* users who patronize facility *n*
$|\mathbf{f}_{mn}(x,y)|$ = the norm of flow vector at $(x,y)\in\Omega$ for class *m* users who patronize facility *n*
$f_{mnx}(x,y)$ = *x* component of the flow vector at $(x,y)\in\Omega$ for class *m* users who patronize facility *n*
$f_{mny}(x,y)$ = *y* component of the flow vector at $(x,y)\in\Omega$ for class *m* users who patronize facility *n*
$c_m(x,y)$ = the transportation cost per unit distance of movement at $(x,y)\in\Omega$ as perceived by class *m* users
$c(x,y)$ = the actual transportation cost per unit distance of movement at $(x,y)\in\Omega$
$a(x,y)$ = the free-flow transportation cost per unit distance at $(x,y)\in\Omega$
$b(x,y)$ = a sensitivity parameter that represents the congestion effect at $(x,y)\in\Omega$
γ_m = a positive conversion factor that transforms the actual cost component to the perceived cost component that is experienced by class *m* users
C_{mn} = the facility cost that is perceived by class *m* users at facility *n*
θ_{mn} = the biased component that represents the preference of class *m* users for facility *n*
$u_{mn}(x,y)$ = the total perceived cost (including transportation and facility costs) at $(x,y)\in\Omega$ for class *m* users who patronize facility *n*
$u_m(x,y)$ = the log-sum cost at $(x,y)\in\Omega$ for class *m* users who patronize all facilities
$q_m(x,y)$ = the total demand rate at $(x,y)\in\Omega$ of class *m* users
$q_{mn}(x,y)$ = the demand rate at $(x,y)\in\Omega$ of class *m* users who patronize facility *n*
Q_{mn} = the total demand of class *m* users who patronize facility *n*
Q_n = the total number of users who patronize facility *n*
ζ_m = the parameter in the logit distribution function for class *m* users

The actual transportation cost per unit distance of travel, which depends on the location and the flow intensities at a particular location, is defined as:

$$c(x,y)=a(x,y)+b(x,y)\sum_{r=1}^{M}\sum_{s=1}^{N}|\mathbf{f}_{rs}(x,y)|,\quad \forall(x,y)\in\Omega\,. \tag{1}$$

This is an isotopic cost function because it depends on the flow intensity but not on the flow direction. This also includes the flow intensities of other classes because there are interactions among different classes of users at a location in the transportation system. Based on the actual transportation cost defined in equation (1), the perceived transportation cost for the class *m* user is defined as:

$$c_m(x,y)=\gamma_m c(x,y)=\gamma_m\left(a(x,y)+b(x,y)\sum_{r=1}^{M}\sum_{s=1}^{N}|\mathbf{f}_{rs}(x,y)|\right),\quad \forall(x,y)\in\Omega, m\in M\,. \tag{2}$$

Users also have to pay for the services at the facility, which is known as the facility cost (or the market externality). The perceived facility cost for class *m* users at facility *n* is, $C_{mn}=\theta_{mn}+\gamma_m S_n(Q_n),\quad \forall n\in N, m\in M$, where $Q_n=\sum_{r=1}^{M}Q_{rn}$ and $S_n(Q_n)$ is the portion of facility cost depending on the usage of that facility. The demand rate is defined as the total number of users per unit area. For each class of users, the demand rate depends on the total cost, which includes the transportation and facility costs, $q_m(x,y) = D_m(u_m(x,y))$, where $D_m(.)$ is the demand function and $u_m(x,y)$ is the log-sum equivalent cost of class *m* users patronizing all facilities,

$$u_m(x,y) = \frac{-1}{\zeta_m} \ln \sum_{s=1}^{N} \exp(-\zeta_m u_{ms}(x,y)), \quad \forall (x,y) \in \Omega . \tag{3}$$

The users are distributed to different facilities based on the following logit-type distribution function:

$$q_{mn}(x,y) = q_m(x,y) \frac{\exp(-\zeta_m u_{mn}(x,y))}{\sum_{s=1}^{N} \exp(-\zeta_m u_{ms}(x,y))}, \quad \forall (x,y) \in \Omega . \tag{4}$$

When the total cost to a particular facility, $u_{mn}(x,y)$, increases, less demand is attracted to this more expensive facility. For each combination of user class and facility, the demand rate and flow vector must satisfy the flow conservation equation, $\nabla \mathbf{f}_{mn}(x,y) - q_{mn}(x,y) = 0, \forall (x,y) \in \Omega$. Assuming that there is no flow across the boundary of the study area, we have the following boundary condition: $\mathbf{f}_{mn}(x,y) = 0, \forall (x,y) \in \Gamma$. However, it is easy to extend the model to represent a given entry or exit demand pattern across the boundary by replacing the above constraint with $\mathbf{f}_{mn}(x,y) \cdot \mathbf{n}(x,y) = g_{mn}(x,y)$, where $g_{mn}(.)$ is the entry or exit demand, and $\mathbf{n}(x,y)$ is the unit normal vector at the boundary. The number of class *m* users who are attracted to facility *n* is obtained as $Q_{mn} = \iint_{\Omega} q_{mn}(x,y) \mathrm{d}\Omega$. From the flow conservation condition around facility *n*, the demand can also be determined by:

$$Q_{mn} - \int_{\Gamma_{nc}} \mathbf{f}_{mn}(x,y) \cdot \mathbf{n}(x,y) \mathrm{d}\Gamma = 0 . \tag{5}$$

3. THE MULTIPLE USER CLASS MODEL

Here we consider a continuum transportation model for which the user equilibrium condition results when users freely make their choice of facilities and their optimal routes within this continuum region based on the criterion of minimizing the total facility and travel cost. This model also includes a logit-type demand distribution function for the probabilistic choice behavior of destinations of the multi-class users and a facility externality that captures the effect of congestion and economies of scale, as the use of the facility varies. This combined distribution and assignment model for multiple user classes in a continuum transportation system, which satisfies the user equilibrium condition and also all of the functional relationships in Section 2, can be formulated as the following minimization problem that was suggested in Ho et al. (2006):

$$\begin{aligned} \underset{\mathbf{f},\mathbf{q},\mathbf{Q}}{\text{Minimize}} \ z(\mathbf{f},\mathbf{q},\mathbf{Q}) = & \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{\theta_{mn} Q_{mn}}{\gamma_m} + \sum_{n=1}^{N} \int_0^{\sum_{m=1}^{M} Q_{mn}} S_n(\xi)\mathrm{d}\xi + \iint_{\Omega} \left\{ \sum_{m=1}^{M}\sum_{n=1}^{N} a|\mathbf{f}_{mn}| + \right. \\ & \frac{b}{2} \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{r=1}^{M}\sum_{s=1}^{N} |\mathbf{f}_{mn}||\mathbf{f}_{rs}| + \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{1}{\zeta_m \gamma_m} (q_{mn} \ln q_{mn} - q_{mn}) - \\ & \left. \sum_{m=1}^{M} \frac{1}{\gamma_m} \int_0^{q_m} D_m^{-1}(\xi)\mathrm{d}\xi - \sum_{m=1}^{M} \frac{1}{\zeta_m \gamma_m} (q_m \ln q_m - q_m) \right\} \mathrm{d}\Omega \end{aligned} \tag{6a}$$

subject to

$$\nabla \mathbf{f}_{mn} - q_{mn} = 0, \quad \forall (x,y) \in \Omega, n \in N, m \in M \ , \tag{6b}$$

$$q_m - \sum_{n=1}^{N} q_{mn} = 0, \quad \forall (x,y) \in \Omega, m \in M \ , \tag{6c}$$

$$\mathbf{f}_{mn} = \mathbf{0}, \quad \forall (x,y) \in \Gamma, n \in N, m \in M \ , \tag{6d}$$

$$\int_{\Gamma_{nc}} \mathbf{f}_{mn} \cdot \mathbf{n} \, \mathrm{d}\Gamma - Q_{mn} = 0, \quad \forall n \in N, m \in M \ . \tag{6e}$$

According to Ho et al. (2006), the following governing equations should be satisfied at the stationary point of the above minimization problem (6):

$$c_m \frac{\mathbf{f}_{mn}}{|\mathbf{f}_{mn}|} + \nabla u_{mn} = 0, \quad \forall (x,y) \in \Omega, n \in N, m \in M \ , \tag{7a}$$

$$\nabla \cdot \mathbf{f}_{mn} - q_{mn} = 0, \quad \forall (x,y) \in \Omega, n \in N, m \in M \ , \tag{7b}$$

$$q_{mn} = q_m \frac{\exp(-\zeta_m u_{mn})}{\sum_{s=1}^{N} \exp(-\zeta_m u_{ms})}, \quad \forall (x,y) \in \Omega, n \in N, m \in M \tag{7c}$$

$$D_m^{-1}(q_m) = -\frac{1}{\zeta_m} \ln \sum_{s=1}^{N} \exp(-\zeta_m u_{ms}), \quad \forall (x,y) \in \Omega, m \in M \tag{7d}$$

$$\mathbf{f}_{mn} = \mathbf{0}, \quad \forall (x,y) \in \Gamma, n \in N, m \in M \ , \tag{7e}$$

$$u_{mn} = \theta_{mn} + \gamma_m S_n(Q_n), \quad \forall (x,y) \in \Gamma_{nc}, n \in N, m \in M \ , \tag{7f}$$

$$\int_{\Gamma_{nc}} \mathbf{f}_{mn} \cdot \mathbf{n} \mathrm{d}\Gamma - Q_{mn} = 0, \quad \forall n \in N, m \in M \ . \tag{7g}$$

Ho et al. (2006) showed that these governing equations satisfy the user equilibrium principle and all the functional relationships discussed in Section 2 for a two-dimensional plane. In the next section, we prove the existence and uniqueness of a solution for this multi-class and multi-facility user equilibrium problem.

4. THE EXISTENCE OF SOLUTION

It is obvious that constraints (6b), (6c), (6d), and (6e) define a feasible region for the objective function to be minimized. Therefore, if there exists one set of solution (**f**, **q**, **Q**) that could simultaneously satisfy the constraints (6b) to (6e) for all combinations of user class and facility, the existence of solution for the minimization problem (6) is proven. This proof will be started by showing the consistence of constraints for one combination of user class and facility. After that, the results of this combination could be further extended to other combinations and thus the existence of solution for the whole problem could be completed.

First, consider the case of class *m* users who patronize facility *n* within the modeled city. In this case, the boundary of other facilities, Γ_{sc}, $\forall s \neq n$, is considered as a part of the boundary Γ. For proving the consistence of the constraints for this combination, the concept of streamlines in fluid mechanics is considered (Douglas et al., 2001; Lowe, 1979). Considering the flow of a steady, incompressible and irrotational ideal fluid within a homogenous region (the modeling city), which takes the boundaries Γ as line sources and the boundary Γ_{nc} as a line sink, there is a family of streamlines between the sources and sinks, as shown in Figure 1. Also, according to the definition of the streamlines, in which a streamline is drawn tangential to the fluid flow direction, there is no flow across streamlines and therefore streamlines do not cross each other. By taking these streamlines as paths that system users travel within the modeled city region, they have the following properties: for each point $(x, y) \in \Omega$ there is a unique path *p*, on which this point lies, from one single boundary point, (x_Γ, y_Γ) to the CBD. Assume that path *p* is represented by the following set of parametric equations, $x = x(t)$ and $y = y(t)$, where $t = 0$ is at point (x_Γ, y_Γ) and $t = 1$ represents the point on the boundary of CBD of this case (i.e. Γ_{nc}). Based on these parametric equations, the following flow pattern is defined as:

$$f_{mnx}(t) = \int_0^t \frac{q_{mn}(x(t), y(t))}{2} \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t \,, \tag{8}$$

$$f_{mny}(t) = \int_0^t \frac{q_{mn}(x(t), y(t))}{2} \frac{\mathrm{d}y}{\mathrm{d}t} \,\mathrm{d}t \,, \tag{9}$$

where q_{mn} could be any smooth arbitrary function within Ω. Differentiating equation (8) with respect to *x*,

$$\frac{\partial}{\partial x} f_{mnx}(t) = \frac{\partial}{\partial x} \int_0^t \frac{q_{mn}(x(t), y(t))}{2} \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t = \frac{\partial}{\partial x} \int_{x_\Gamma}^x \frac{q_{mn}(x, y)}{2} \,\mathrm{d}x = \frac{q_{mn}}{2} \,. \tag{10}$$

Similarly, differentiating equation (9) with respect to *y*, it gives:

$$\frac{\partial f_{mny}}{\partial y} = \frac{q_{mn}}{2} \,. \tag{11}$$

Substituting equations (10) and (11) into the left-hand side of constraint (6b) for class *m* users who patronize facility *n*, we have:

$$\frac{\partial f_{mnx}}{\partial x} + \frac{\partial f_{mny}}{\partial y} - q_{mn} = \frac{q_{mn}}{2} + \frac{q_{mn}}{2} - q_{mn} = 0 \,. \tag{12}$$

Thus, this definition of flow pattern satisfies constraint (6b). Considering the value of the flow components at the point (x_Γ, y_Γ), by equations (8) and (9) we have:

$$f_{mnx}(0) = \int_0^0 \frac{q_{mn}(t)}{2} \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t = 0 \,, \tag{13}$$

$$f_{mny}(0) = \int_0^0 \frac{q_{mn}(t)}{2} \frac{\mathrm{d}y}{\mathrm{d}t} \,\mathrm{d}t = 0 \,. \tag{14}$$

Combining equations (13) and (14) gives constraint (6d). The assumed flow pattern can be easily extended to the more general case of constraint (6d) (i.e. $\mathbf{f}_{mn} \cdot \mathbf{n} = g_{mn}$) by adding a constant term $g_{mn}/2n_x$ and $g_{mn}/2n_y$ to equations (8) and (9) respectively, where n_x and n_y are the *x* and *y* component of the unit normal vector **n**. Constraints (6c) and (6e) are actually the definition of variables q_m and Q_{mn} respectively. As Q_{mn} only appears in constraint (6e) and q_m only in (6c), by assuming:

$$Q_{mn} = \int_{\Gamma_{nc}} \mathbf{f}_{mn} \cdot \mathbf{n} \,\mathrm{d}\Gamma \,, \tag{15}$$

and

$$q_m = \sum_{s=1}^N q_{ms} \,, \tag{16}$$

constraints (6e) and (6c) will be satisfied. With equations (8), (9), (15), (16) and a continuously defined q_{mn}, the consistency of constraints for this combination of user class and facility is proved. By extending this to other combinations, the proof of the existence of the solution for the minimization problem (6) is completed.

5. UNIQUENESS OF SOLUTION

The uniqueness of the solution for the minimization problem (6) is analyzed in this section. The method that is used is based on that introduced by Beckmann (1952).

Proposition: For a strictly decreasing elastic demand function $D_m(u_m(x,y))$, a strictly increasing facility cost function C_{mn}, and a strictly increasing perceived transportation cost function $c_m(x,y)$, the solution that satisfies the governing equations (7a) to (7g), which solves the minimization problem (6), is uniquely defined.

Proof: Assume that $\boldsymbol{\Psi}_1 = (\mathbf{f}_{1x}, \mathbf{f}_{1y}, \mathbf{u}_1)$ and $\boldsymbol{\Psi}_2 = (\mathbf{f}_{2x}, \mathbf{f}_{2y}, \mathbf{u}_2)$ are two different sets of solutions for minimization problem (6) that satisfies equations (7a) to (7g), considering the following equation:

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega} \nabla\cdot\left[(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\right]\mathrm{d}\Omega$$
$$=\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{nc}}(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\mathbf{n}\ \mathrm{d}\Gamma+\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma}(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\mathbf{n}\ \mathrm{d}\Gamma.$$

This equation is based on Green's theorem in the two-dimensional case; thus, it should always be satisfied for any assumed sets of solutions $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$. After expanding and rearranging it gives:

$$\begin{aligned}\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma}(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\mathbf{n}\ \mathrm{d}\Gamma &= \sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}(u_{1mn}-u_{2mn})\nabla\cdot(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\,\mathrm{d}\Omega \\ &+\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\nabla(u_{1mn}-u_{2mn})\,\mathrm{d}\Omega \\ &-\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{nc}}(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\mathbf{n}\ \mathrm{d}\Gamma\ .\end{aligned} \tag{17}$$

Equation (17) will be used as the basis for proving the uniqueness of the solution of this multi-class user equilibrium problem in a continuum transportation system. Considering the left-hand side of equation (17), by equation (7e) (i.e. $\mathbf{f}_{1mn} = \mathbf{f}_{2mn} = 0$ on Γ, $\forall$ m and n) we have:

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma}(u_{1mn}-u_{2mn})(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\mathbf{n}\ \mathrm{d}\Gamma = 0\ . \tag{18}$$

This indicates that the left-hand side of equation (17) will vanish if both $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ are the solution of the minimization problem (6). Consider the case of class m users who patronize facility n. Applying equation (7b) to the first term on the right-hand side of equation (17) gives:

$$\iint_{\Omega}(u_{1mn}-u_{2mn})\nabla\cdot(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\,\mathrm{d}\Omega=\iint_{\Omega}(u_{1mn}-u_{2mn})(q_{1mn}-q_{2mn})\,\mathrm{d}\Omega\,. \tag{19}$$

Combining Equation (7c) with (7d) and applying the definition of the log-sum cost (u_m), we have,

$$q_{mn}=D_m(u_m)\left(1-\frac{\sum_{s=1,s\neq n}^{N}\exp(-\zeta_m u_{ms})}{\sum_{s=1}^{N}\exp(-\zeta_m u_{ms})}\right). \tag{20}$$

As the log-sum cost (u_m) is a strictly increasing function of the travel cost u_{mn} and the elastic demand function $D_m(u_m)$ is taken as a strictly decreasing function with respect to the log-sum cost, the demand q_{mn} will decrease when cost u_{mn} increases. Thus, it can be concluded that for any pair of different solutions $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$, equation (19) satisfies the following:

$$\iint_{\Omega}(u_{1mn}-u_{2mn})\nabla\cdot(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\,\mathrm{d}\Omega=\iint_{\Omega}(u_{1mn}-u_{2mn})(q_{1mn}-q_{2mn})\,\mathrm{d}\Omega<0\,.$$

Summing up for all combinations of user class and facility,

$$\begin{aligned}&\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}(u_{1mn}-u_{2mn})\nabla\cdot(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\,\mathrm{d}\Omega\\&=\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}(u_{1mn}-u_{2mn})(q_{1mn}-q_{2mn})\,\mathrm{d}\Omega<0\,.\end{aligned} \tag{21a}$$

Consider the combination of class m user and facility n. Applying equation (7a) to the second term on the right-hand side of equation (17) gives:

$$\begin{aligned}&\iint_{\Omega}(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\nabla(u_{1mn}-u_{2mn})\,\mathrm{d}\Omega\\&=\iint_{\Omega}(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\left(c_{2m}\frac{\mathbf{f}_{2mn}}{|\mathbf{f}_{2mn}|}-c_{1m}\frac{\mathbf{f}_{1mn}}{|\mathbf{f}_{1mn}|}\right)\mathrm{d}\Omega\\&=\iint_{\Omega}-c_{1m}|\mathbf{f}_{1mn}|-c_{2m}|\mathbf{f}_{2mn}|+\frac{\mathbf{f}_{1mn}\cdot\mathbf{f}_{2mn}}{|\mathbf{f}_{1mn}||\mathbf{f}_{2mn}|}\left(c_{1m}|\mathbf{f}_{2mn}|+c_{2m}|\mathbf{f}_{1mn}|\right)\mathrm{d}\Omega\\&\leq\iint_{\Omega}-c_{1m}|\mathbf{f}_{1mn}|-c_{2m}|\mathbf{f}_{2mn}|+c_{1m}|\mathbf{f}_{2mn}|+c_{2m}|\mathbf{f}_{1mn}|\,\mathrm{d}\Omega\\&=\iint_{\Omega}-c_{1m}\left(|\mathbf{f}_{1mn}|-|\mathbf{f}_{2mn}|\right)-c_{2m}\left(|\mathbf{f}_{2mn}|-|\mathbf{f}_{1mn}|\right)\mathrm{d}\Omega\\&=\iint_{\Omega}(c_{1m}-c_{2m})\left(|\mathbf{f}_{2mn}|-|\mathbf{f}_{1mn}|\right)\mathrm{d}\Omega\,.\end{aligned}$$

Therefore, as the transportation cost function c_m is assumed to be strictly increasing with the flow intensity $|\mathbf{f}_{mn}|$, the second term on the right-hand side of equation (17) satisfies the following inequality for any pair of different solutions $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$:

$$\iint_{\Omega}(\mathbf{f}_{1mn}-\mathbf{f}_{2mn})\cdot\nabla(u_{1mn}-u_{2mn})\,\mathrm{d}\Omega\leq\iint_{\Omega}(c_{1m}-c_{2m})\left(|\mathbf{f}_{2mn}|-|\mathbf{f}_{1mn}|\right)\mathrm{d}\Omega<0\,.$$

Summing up for all combinations of user class and facility,

$$
\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\nabla\left(u_{1mn}-u_{2mn}\right)\mathrm{d}\Omega \\
&\leq \sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(c_{1m}-c_{2m}\right)\left(\left|\mathbf{f}_{2mn}\right|-\left|\mathbf{f}_{1mn}\right|\right)\mathrm{d}\Omega<0\,.
\end{aligned}
\tag{21b}
$$

Similarly, consider the combination of class m user and facility n. Applying equations (7f) and (7g) to the third term on the right-hand side of equation (17) gives:

$$
\begin{aligned}
-\int_{\Gamma_{nc}}\left(u_{1mn}-u_{2mn}\right)\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\mathbf{n}\ \mathrm{d}\Gamma &= -\left(C_{1mn}-C_{2mn}\right)\int_{\Gamma_{nc}}\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\mathbf{n}\ \mathrm{d}\Gamma \\
&= \left(C_{1mn}-C_{2mn}\right)\left(Q_{2mn}-Q_{1mn}\right).
\end{aligned}
$$

Therefore, under the assumption that the facility cost function C_{mn} is a strictly increasing function with the usage Q_{mn}, the third term on the right-hand side of equation (17) satisfies the following inequality for any pair of different solutions $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$:

$$
-\int_{\Gamma_{c}}\left(u_{1mn}-u_{2mn}\right)\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\mathbf{n}\ \mathrm{d}\Gamma = \left(C_{1mn}-C_{2mn}\right)\left(Q_{2mn}-Q_{1mn}\right)<0\,.
$$

For all combinations of user classes and facility, we have:

$$
\begin{aligned}
&-\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{c}}\left(u_{1mn}-u_{2mn}\right)\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\mathbf{n}\ \mathrm{d}\Gamma \\
&= \sum_{m=1}^{M}\sum_{n=1}^{N}\left(C_{1mn}-C_{2mn}\right)\left(Q_{2mn}-Q_{1mn}\right)<0.
\end{aligned}
\tag{21c}
$$

Substituting the results of equations (21a), (21b), and (21c) to the right-hand side of equation (17), it becomes:

$$
\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(u_{1mn}-u_{2mn}\right)\nabla\cdot\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\mathrm{d}\Omega+\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\nabla\left(u_{1mn}-u_{2mn}\right)\mathrm{d}\Omega \\
&\quad-\sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{nc}}\left(u_{1mn}-u_{2mn}\right)\left(\mathbf{f}_{1mn}-\mathbf{f}_{2mn}\right)\cdot\mathbf{n}\ \mathrm{d}\Gamma \\
&\leq \sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(u_{1mn}-u_{2mn}\right)\left(q_{1mn}-q_{2mn}\right)\mathrm{d}\Omega+\sum_{m=1}^{M}\sum_{n=1}^{N}\iint_{\Omega}\left(c_{1m}-c_{2m}\right)\left(\left|\mathbf{f}_{2mn}\right|-\left|\mathbf{f}_{1mn}\right|\right)\mathrm{d}\Omega \\
&\quad+\sum_{m=1}^{M}\sum_{n=1}^{N}\left(C_{1mn}-C_{2mn}\right)\left(Q_{2mn}-Q_{1mn}\right) \\
&<0\,.
\end{aligned}
\tag{22}
$$

This contradicts the result in equation (18) for the left-hand side of equation (17). As equation (17) should always be satisfied, the right-hand side of equation (17) should vanish. Thus, by considering equations (21a), (21b) and (21c), $\mathbf{\Psi}_1$ should be identical to $\mathbf{\Psi}_2$, and this completes the proof.

6. CONCLUSIONS

This paper proves the existence and uniqueness of the solution for the multi-class user equilibrium problem in a continuum transportation system that was introduced by Ho et al. (2006). In the proof of the existence of the solution, the concept of streamlines from fluid mechanics is adopted to show the existence of paths with specific characteristics. The uniqueness of solution requires a strictly increasing facility cost function, a strictly decreasing elastic demand function, and a strictly increasing transportation cost function. This study provides a solid theoretical foundation for the future applications of the multi-class continuum transportation model, for example the facility competition problems, such as airport competition and coordination (Loo et al., 2005), the cordon-based congestion pricing problems (Ho et al., 2005; Hau, 2005a,b, 2006a,b), and the housing location problem (Ho and Wong, 2005, 2007).

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