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# USE OF TRAVEL DEMAND SATISFACTION TO ASSESS ROAD NETWORK RELIABILITY

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A new concept of network reliability is introduced to assess the performance of the road network. It is referred to as the travel demand satisfaction reliability (DSR). In order to understand this new concept, the attributes to recurrent travel demand are firstly investigated for the analysis of road network reliability. The latent travel demand (LTD) is then distinguished and estimated by elastic travel demand functions. The ratio of the equilibrium travel demand and LTD can be used to ascertain the extent of satisfying and revealing the latent travel demand (DSI), which is affected by both the uncertainty of the network degradation and of the recurrent travel demand. The DSR is defined as the probability that the road network can accommodate a given DSI. The DSR can be extended to include other reliability measures such as travel time reliability under certain conditions. In order to facilitate the presentation of the essential ideas, simple examples are employed for discussion together with mathematical formulation.

KEYWORDS: Network reliability, demand uncertainty, degradable network, network equilibrium, sensitivity analysis

## 1. INTRODUCTION

Travel demand is a derived demand because people are required to make trips to satisfy their desires or needs (Lam and Yin, 2001; Joh et al., 2006). In general, the road systems should be designed to cater for the *recurrent travel demand* (RTD) that results from a range of normal and regular special events. The normal events, such as the growth of economic and non-work activities during peak periods, will add more traffic onto the road network. On the other hand, some of the regular special events, such sporting and weekend events, will also add a lot of vehicle trips to the road network for recreational activities. All of these events are referred to as *latent demand event* (LDE). However, it should be pointed out that the irregular and/or non-recurrent special events, such as Olympic Games, are not considered in this paper.

Despite the normal and regular special events, the LDE can be further categorised by the long-term impact and short-term effect on the variation of the RTD. Dowling and Colman (1995) reckoned that the long-term impact were attributed by the factors of growing local economy, the improvement of natural constraints and environment, the increased income, population and employment. It was also caused by deterioration of transit service, high transit fares, changes in taste, automobile ownership and land use. The short-term effects were due to the factors of re-scheduling of the trip; the choice of the routes taken; the choice of the travel modes used (e.g. carpool, transit, drive alone), the frequency of the trip made and the linking activities of the journey.

Lam and Yin (2001) revealed that the utility of traveller activities was the kernel factor of making trips. Suppose that there is a new road built or a road expansion in the commuters' corridor. The new transport environment will induce the change of activity patterns and traffic flow distribution by the interaction between activity/travel behaviour.

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In the morning-peak period, some commuters may depart home for work later than before because of the shorter journey time due to the road expansion scheme. Some commuters may depart home earlier for a non-work activity during the commuting, such as stopping at bank or driving their children to school on the way to work, which he has never done before due to the tighter time budget under the condition without road improvement. It is believed that the greater the utility of the non-work activity (i.e. the larger the benefit of pursuing the activity), the higher is the probability of making trips for satisfying these non-work activities. As a result, the origin-destination (OD) trips will be increased and changed temporally and spatially.

During the last decade, more and more transportation professionals have realised that peak-period congestion cannot be caused entirely by the home-to-work or work-to-home trips. The growing number of non-work trips made by individuals in the morning and evening peak periods substantially contributes to road congestion (Bhat, 1997). To save time, more and more people tend to chain trips together. Crevo and Virkud (1994) reported that the peak-hour traffic could include trips for various purposes, such as work, shop, home, school, goods movement and others. Yai et al. (1995) also showed that the recreational activities would produce heavy traffic volumes and hence create congestion on some roads in suburban areas. The average distance of recreational trips is twice as long as that of commuter trips, and the total vehicle kilometres travelled per day of week-end recreational trips is much higher than that of commuter trips. As we experienced, the recreational activities could lead to more travels and road congestion in downtown area particularly on Friday.

In view of the above, the RTD is in fact a random variable and should be considered as a travel demand function of the travel time by OD pair and of the LDE. The actual total trips or the total demands are generated by equilibrium of the RTD and network supply. The need and/or desire of making trips would be enlarged by these events (LDE), and so does the total demand. If there is no resource limitation, the road system should be designed for catering the *latent travel demand* (i.e. the unconstrained scenario). However, if the network cannot satisfy the total need of all travellers or the *latent travel demand* (LTD), some of these travellers will be suppressed, postponed or changed to use other modes of transport due to the road congestion and the network capacity constraint (i.e. the constrained scenario). The resultant demand is then the RTD. We use the demand generated under free-flow condition as the reference for these measures because this provides an achievable level. Although this level of demand is subject to the change of the transportation network, it provides a suitable reference for assessment of the network performance.

In principle, the traffic equilibrium is derived from the interaction of the RTD and network supply. On the demand side, the RTD can be influenced by a number of factors (or attributes) such as the travel time by OD pair and the need of making the travel (i.e. LDE). On the supply side, the road network should be designed to cater for the needs of travel. However in the analysis of the network equilibrium, it was always assumed that the OD flows are the function of only one variable such as the travel time by OD pair (see LeBlanc and Farhangian, 1981; Dafermos, 1982; Du and Nicholson, 1997; and Asakura, 1998). In other words, these related studies assumed that all the parameters of the demand function are fixed. From the viewpoint of economics (see Mankiw, 1998), if the demand function is expressed as a function with exclusive variable such as travel time, then there must be a *positive constant* item to reflect the impacts of LDE on the RTD in the traditional demand function. As a result of the LDE at a given travel time by OD pair, the RTD is increased, and so, such constant term should be larger for the unconstrained scenario than that for the constrained scenario. The demand curve for the unconstrained scenario should be shifted to the right as compared to the one for the constrained scenario. Dowling and Colman (1995) discussed the effects of increased highway capacity on the shift of travel demand curve. In this paper, a new reliability measure (or performance indicator) is proposed to assess the reliabilities of degradable road network for satisfying and/or revealing the latent travel demand.

Reliability has long been a key topic in management and operation studies as well as system engineering. The importance of reliability as an element of customer satisfaction and product differentiation in all industries cannot be overstated. In general, reliability is defined as the ability or capability of product to perform the specified function in the designated environment for a minimum length of time or minimum number of cycles or events (see Ireson et al., 1996). The reliability in system networks, such as communication network, is defined as the probability that the maximal flow can be assigned to the communication network (see Ball et al., 1995). Basically, this is a system optimisation (SO) problem as the system manager can allocate the communication flow at his own wish. However, such allocation cannot be applicable directly to road network as the road users would make their own travel choices including trip generation, modal split, trip distribution and traffic assignment in view of the road traffic conditions. Therefore, the reliability of road network is different from the system reliability and should be considered for road network design. It is important to take into account the change of travel demand and route choice behaviour for assessing the performance of the alternative road networks.

In recent years the road network reliabilities have been considered as a new avenue for future transportation research in terms of both theories and practical applications. Nicholson and Du (1997) highlighted the importance of studying the reliability of degradable transportation system as a lifeline. Iida (1999) echoed that road users in the 21$^{st}$ century would prefer to a more reliable transport system where they can be confident of arriving at their destination on schedule. Road network reliability analysis will play an important role in the planning, design and management of road networks particularly in developed countries and/or megacities. Bell (1999) stated that the potential sources of disruption to transport networks are numerous, from natural or man-made disasters to peak period congestion. Thus it is of interest to design transport networks to minimise the disruption due to events of these kinds, as any breakdown of the transport network in modern cities can cause substantial economic losses and social unrest. There is therefore a need to define the concept of transport network reliability for assessing the network performance (Lam, 1999).

### *1.1 Measures reliability in road networks*

The previous proposed definitions of network reliability can be classified into four aspects: (i) Connectivity in degraded transport networks (see Iida and Wakayabashi, 1989); (ii) Travel time reliability due to fluctuation of link flow (see Asakura and Kashiwadani, 1991) and capacity degradation (see Asakura, 1998); (iii) The reliability of decreased traffic flow in degraded transport system (see Du and Nicholson 1997); (iv) Network reserve capacity in degraded network (see Chen, et al. 1999). The principal characteristics of such definitions are briefly discussed below and summarised in Table 1.

TABLE 1: Principal characteristics of four definitions of road network reliability

| Reliability | Performance indicator | Uncertainty | Constraints | Probability definition |
|---|---|---|---|---|
| Connectivity (see Iida and Wakayabashi, 1989) | $\theta_C = 1$ if connected and $\theta_C = 0$ if disconnected | Disruption of road links | Not included | connected and disconnected network |
| Travel time (I) (see Asakura and Kashiwadani, 1991) | Specified travel time $\theta_T$ | Fluctuation of daily traffic flow | Constant demand and no link capacity | Travel time less than $\theta_T$ |
| Travel time (II) (see Asakura, 1998) | Specified network service level $\theta_L$ | Degradable link capacity | Constant or elastic demand function and link capacity constraint | Service level less than $\theta_L$ |
| System and OD sub-system (see Du and Nicholson, 1997) | Intolerable decrement rate of OD flow $\theta_O$ | Degradable link capacity | Elastic demand function and link capacity constraint | Decrement rate less than $\theta_O$ |
| Capacity (see Chen et al., 1999) | Required demand level $\theta_D$ | Degradable link capacity | Proportional OD trip table and link capacity constraint | Network reserve capacity greater than $\theta_D$ |

First, *connectivity* reliability is stated as a probability of the ability to reach a given destination from an origin (see Iida and Wakayabashi, 1989). There exist at least one path without disruption or heavy delay to a given destination within a given time period. It reflects intuitively the topological accessibility of trips to a destination from an origin. It is also concerned with the possibility that the zone centroids or nodes of network remain connected. The earliest studies of connectivity reliability are in the vein of systems reliability. The performance measure of connectivity reliability, denoted as $\theta_C$, is a binary state, with values for connected ($\theta_C = 1$) and disconnected ($\theta_C = 0$).

Second, *travel time* reliability is defined as the probability of the ability to reach a given destination from an origin within a given travel time ($\theta_T$) (see Asakura and Kashiwadani, 1991). The primary concern is whether a driver can arrive at his terminal on time when the link travel time is a random variable. The link travel time, as a result of the daily traffic variation, was usually assumed with Normal distribution. Asakura (1998) proposed a model for evaluating travel time reliability between an OD pair in a network deteriorated by natural disasters. The performance measure of travel time reliability, denoted as $\theta_L$ and referred as to network service level, is the ratio of travel times under the degraded and non-degraded states. It implies the requirement of $\theta_L \geq 1.0$. The travel time reliability is the probability that the current level of network service is less than the specified network service level (i.e. an acceptable level of travel time). The specified network service level must be given and larger than 1.0. Otherwise the reliability is always 0% while the network is degraded due to $\theta_L \geq 1.0$. This definition of travel time reliability can be extended to include the connectivity reliability as the value of $\theta_L$ increases without bound.

Third, Du and Nicholson (1997) defined a probability that the decrement rate of OD flow is less than a given intolerable value as the reduction of travel demand is due to the degradable network. Two definitions; namely, *system and OD sub-system* reliabilities are given. They are related to the reduction of travel demand (system-wide and OD pair) due to the degradable network. A system surplus was suggested as a performance measure to assess the socio-economic impacts of the system degradation. The decrement rate of travel demand was used as the performance indicator $\theta_O$, which is the ratio of the difference between the non-degradable and degradable OD flows. It implies that the constraint falls in the range of $0 \le \theta_O \le 1.0$. The reliability is always 100% when the intolerable value is greater than 1.0.

Lastly, Chen et al. (1999) extended the reserve capacity concept for traffic signalized intersection to that for the whole network, so-called the reserve network capacity reliability. The latter was considered as an indicator $\theta_D$ for measuring the demand levels that can be revealed and/or satisfied by the road network. Thus, the *capacity* reliability was defined as the probability that the demand level under the degraded network condition is greater than the specified demand level. In other words, it was related to the network capacity under degradation that can accommodate a certain demand level. The capacity reliability is a function of the specified demand level. If the network capacities are given, it is determined whether the network can or cannot accommodate the specified demand level. It implied that if the network capacities were not assumed as a random variable, the reliability should be a binary state. The capacity reliability is 0 when the network cannot accommodate the specified demand level. Otherwise the capacity reliability is 1 when the network can accommodate the specified demand level. The introduction of a continuous network reliability index between 0 and 1 was given in Chootinan et al. (2005).

As summarised in Table 1 and discussed as above, the connectivity reliability is measured by the probability of the 0-1 distribution but the others are measured by the probability that the indicator of performance is less or more than a given value. For determining the uncertainty, the travel time (II), system and OD sub-system and capacity reliabilities are all subject to the random degradable network, and yet the travel time (I) reliability is dependent on the random variation of link time. For forecasting the OD flows, the fixed travel demand functions are set as the input in modelling the travel time (I and II), system and OD sub-system reliabilities. For the analysis of capacity reliability, the input of OD flows is the maximal OD flows that a network can accommodate a given OD trip table times an arbitrary multiplier. The maximal OD flows can be referred to as the constrained travel demand depending on the network capacity. It is observed from the above that the RTD and its stochastic increment from the LDE have not yet been considered in the previous related studies.

*1.2 The recurrent travel demand issues*

When the LDE occurs, the road network satisfies some of the travel demands but the excess demand is suppressed, postponed or changed to use other modes of transport. The equilibrium travel demand (ETD) is the **satisfied or revealed** travel demand that can be satisfied by using the road network for their travel as requested. The **non-satisfied** or non-revealed travel demand (NTD) is the one that intends to use the road network but cannot be satisfied or revealed due to the road congestion and the network capacity constraint. The sum of the ETD and NTD is denoted as the *latent travel demand* (LTD). The LTD is thus the total travel demand (by OD pair) that intends to use the road network with reference to a specified traffic condition such as the free-flow travel time by OD pair.

In short, the RTD by OD pair is a random variable, which is a function of both the OD travel time and LDE. The ETD is the resultant RTD when the LDE occurs under network capacity constrained condition. In other words, the travel demand is at the equilibrium of the RTD and the network supply constraint. On the other hand, the LTD becomes the RTD when the LDE occurs under unconstrained circumstances and the road traffic is under the free-flow condition in the network. Both the ETD and LTD are the derived demand from LDE when travel times are given and/or specified. Therefore, both the ETD and LTD are derived from the LDE.

In this paper, the ratio of the equilibrium and latent travel demand is suggested as the performance indicator ( $\theta_S$ ), referred to as the *travel demand satisfaction ratio* (DSI). It reflects the ability of the network for satisfying the latent travel demand. Therefore, the *travel demand satisfaction reliability* (DSR) is proposed for analysis of how a network satisfying and revealing the latent travel demand. The DSR is defined as the probabilities that the DSI is not less than a certain specified value. Without taking into account the effects of the LDE, the definition of DSR is indifferent in nature to the other reliability measures such as the travel time, system and OD sub-system reliabilities. However, once the LDE occurs, the reliability measure is generally **over-estimated** if the definitions of the travel time, system and OD sub-system reliabilities are adopted, and is always **under-estimated** by the definition of the capacity reliability. These over-estimation and under-estimation results will be explained and illustrated by a simple example as shown later in this paper.

Both the DSR and capacity reliability are focused on the ability of the network satisfying the travel demand. However, it should be noted that capacity reliability is a function of the link capacities and the network reserve capacity is the multiplier times a given OD trip matrix. Traffic queues are not allowed in the estimation of reserve capacity as the link capacity constraint has been applied. The given OD trip matrix is referred to as an unchangeable recurrent travel demand. Therefore, the capacity reliability can only consider the uncertainty of the reserve capacity while the recurrent travel demand is assumed constant and fixed. On the other hand, the proposed DSR would take into account the variation of the recurrent travel demand for evaluating the performance of the road network.

In order to assess the DSR, the travel demand function should be included for analysis and it should be elastic so as to reflect the phenomena in our real world. Note that the capacity constraints on road links in the network can be either relaxed or imposed so that traffic queues are allowed for analysis of network reliability. In this paper, the equilibrium travel demand or OD flow is assigned to road network under the assumption of user equilibrium (UE) behaviour. The variation of OD flows may be due to the occurrence of LDE or degradation of network or both together. As a result from the exogenous uncertainty, the DSI is also a random variable. The DSR can be evaluated in confrontation with the uncertainties of both the LDE and degradable network. In this paper, it is shown that the DSR can be equivalent to the travel time reliability and OD sub-system reliability under certain conditions and/or special cases.

This paper scrutinises the concept of reliability of road network in a broader context, in which various measures are investigated to assess the reliabilities of road network for satisfying and revealing the latent travel demand. In particular, we introduce a novel measure the proportion of the latent demand that is satisfied and revealed at equilibrium.

We establish its properties and investigate its response to a set of test variations in each of supply (through reductions in network capacity) and demand (through increases in the latent travel demand) in a small example network. We pay special attention to the interaction between these two kinds of variation. We develop a linear sensitivity analysis and show that this is accurate for small perturbations and can be used together with knowledge of the distribution of perturbations to the parameters. However, as would be expected, the linear sensitivity analysis does not capture the full response for larger variations. In particular, we show that because the sensitivity analysis is first order, it cannot capture the interactions between supply and demand variations, which can be important in some cases.

## 2. ANALYSIS OF TRANSPORT SYSTEMS IN EQUILIBRIUM

### *2.1 Introduction*

In the present context of network reliability in response to variations in supply and demand that can be anticipated, we suppose that the system is in equilibrium. This extends to both the travel patterns within the network, and the relationship between the demand for travel and the travel time at which it can be achieved. The latter of these corresponds to an equilibrium between the demand for travel, reflecting both regular and occasional elements, and the travel facility supplied by the network, reflecting its normal capacity as modified by any reductions that could be anticipated. The equilibrium travel patterns in the network govern the relationship between the amount of travel demand that takes place and the travel times incurred.

In this section, we introduce the form of the demand and network models that will be adopted. On the one hand, the demand model can be analysed directly, but on the other hand the network model cannot usually, because it includes its own element of response through route choice. Because of this, we develop a sensitivity analysis for the effects of variations in supply and demand on the operation of the network in equilibrium and consider how this can be used to estimate the effects of these variations in measures of network performance.

### *2.2 The demand model*

We suppose that for each origin-destination pair $o\text{-}d$, the demand for travel $q_{od}$ depends on an appropriate measure $G_{od}$ of the cost of travel in a way that is controlled parametrically by some vector $\boldsymbol{\varphi}$ of relevant quantities. Thus we represent formally in vector form the demand function as $\mathbf{q}(\mathbf{G}, \boldsymbol{\varphi})$. For the purposes of the present analysis it is more convenient to express the resulting relationship between travel demand and cost in the form of the inverse demand function which is here denoted as $\mathbf{p}(\mathbf{q}, \boldsymbol{\varphi})$ that satisfies for any origin-destination flows $\hat{\mathbf{q}}$ the relationship $\hat{\mathbf{q}} = \mathbf{q}\left[\mathbf{p}(\hat{\mathbf{q}}, \boldsymbol{\varphi}), \boldsymbol{\varphi}\right]$. This inverse demand function indicates the costs at which a specified pattern of travel demand will occur and hence relates to the equilibrium travel demand. Variations in the level of demand and in the form of the demand function itself, and hence in the inverse demand function, can be represented through appropriate use of the parameters $\boldsymbol{\varphi}$.

In order to undertake a sensitivity analysis of demand-supply equilibrium, we will need to make use of the matrices of partial derivatives of the inverse demand functions $\mathbf{p}$ with respect to each of the origin-destination flows $\mathbf{q}$ and the parameters $\boldsymbol{\varphi}$. Thus we denote $\mathbf{P}$ as the matrix of derivatives of the vector of origin-destination inverse demands with respect to the origin-destination flows $\mathbf{q}$ that has elements $P_{od,rs} = \partial p_{od} / \partial q_{rs}$, and $\mathbf{P}^{\varphi}$ as the matrix of derivatives of the vector of vector of origin-destination inverse demands with respect to the parameters $\boldsymbol{\varphi}$ that has elements $P^{\varphi}_{od,i} = \partial p_{od} / \partial \varphi_i$.

By way of illustration, we consider two examples of demand functions: the power law and the exponential demand function.

### *2.2.1 Example 1: the power law*

In this case, the demand function can be expressed in the form

$$q_{od}\left(\mathbf{G}, \boldsymbol{\varphi}\right) = \phi^0_{od}\, G_{od}^{-\boldsymbol{\varphi}^1_{od}} .$$

For the sake of clarity, we suppress the dependence on the origin-destination pair *o-d*. Differentiating this function with respect to its parameters yields

$$\frac{\partial q}{\partial \boldsymbol{\varphi}^0} = C^{-\boldsymbol{\varphi}^1} = \frac{q\left(\mathbf{G}, \boldsymbol{\varphi}\right)}{\boldsymbol{\varphi}^0},$$

$$\frac{\partial q}{\partial \boldsymbol{\varphi}^1} = -\log_e\left(C\right) \boldsymbol{\varphi}^0\, G^{-\boldsymbol{\varphi}^1} = -\log_e\left(G\right) q\left(\mathbf{G}, \boldsymbol{\varphi}\right).$$

The derivative with respect to the cost of travel is

$$\frac{\partial q}{\partial G} = -\boldsymbol{\varphi}^1\, \boldsymbol{\varphi}^0\, C^{-\left(\boldsymbol{\varphi}^1 + 1\right)} = \frac{-\boldsymbol{\varphi}^1\, q\left(\mathbf{G}, \boldsymbol{\varphi}\right)}{G} .$$

The point cost elasticity $\varepsilon_c$ of demand is defined as $\varepsilon_G = \dfrac{\mathrm{d}\,q}{\mathrm{d}\,G}\dfrac{G}{q}$ so that in this case is constant and determined by the parameter $\varepsilon_G = -\boldsymbol{\varphi}^1$.

This function can be inverted readily to give the inverse demand function for this case as

$$p\left(\mathbf{q}, \boldsymbol{\varphi}\right) = \left(\frac{\boldsymbol{\varphi}^0}{q}\right)^{1/\boldsymbol{\varphi}^1},$$

which has derivatives with respect to the parameters

$$\frac{\partial p}{\partial \boldsymbol{\varphi}^0} = \frac{1}{\boldsymbol{\varphi}^0\, \boldsymbol{\varphi}^1}\left(\frac{\boldsymbol{\varphi}^0}{q}\right)^{1/\boldsymbol{\varphi}^1} = \frac{p\left(\mathbf{q}, \boldsymbol{\varphi}\right)}{\boldsymbol{\varphi}^0\, \boldsymbol{\varphi}^1},$$

$$\frac{\partial p}{\partial \boldsymbol{\varphi}^1} = \log_e\left(\frac{q}{\boldsymbol{\varphi}^0}\right) \frac{p\left(\mathbf{q}, \boldsymbol{\varphi}\right)}{\left(\boldsymbol{\varphi}^1\right)^2},$$

and derivative with respect to demand that is

$$\frac{\partial p}{\partial q} = \frac{-1}{\boldsymbol{\varphi}^1} \frac{\left(\boldsymbol{\varphi}^0\right)^{\left(1/\boldsymbol{\varphi}^1\right)}}{q^{\left(1/\boldsymbol{\varphi}^1 - 1\right)}} = \frac{-\,p\left(\mathbf{q},\boldsymbol{\varphi}\right)}{\boldsymbol{\varphi}^1\, q}$$

*2.2.2 Example 2: the exponential demand function.*

In this case, the demand function can be expressed in the form:

$$q_{od}\left(\mathbf{G},\boldsymbol{\varphi}\right) = \boldsymbol{\varphi}^0_{od}\, e^{-\boldsymbol{\varphi}^1_{od}\, G_{od}}$$

Differentiating this function with respect to its parameters (again suppressing dependence on the origin-destination pair) yields

$$\frac{\partial q}{\partial \boldsymbol{\varphi}^0} = e^{-\boldsymbol{\varphi}^1 G} = \frac{q\left(\mathbf{G},\boldsymbol{\varphi}\right)}{\boldsymbol{\varphi}^0},$$

$$\frac{\partial q}{\partial \boldsymbol{\varphi}^1} = -G\, \boldsymbol{\varphi}^0\, e^{-\boldsymbol{\varphi}^1 G} = -G\, q\left(\mathbf{G},\boldsymbol{\varphi}\right).$$

The derivative with respect to the cost of travel is

$$\frac{\partial q}{\partial G} = -\boldsymbol{\varphi}^1\, \boldsymbol{\varphi}^0\, e^{-\boldsymbol{\varphi}^1 G} = -\boldsymbol{\varphi}^1\, q\left(\mathbf{G},\boldsymbol{\varphi}\right),$$

so that the point cost elasticity in this case is $\varepsilon_G = -\boldsymbol{\varphi}^1 G$.

The inverse demand function for this exponential demand function is

$$p\left(\mathbf{q},\boldsymbol{\varphi}\right) = \frac{1}{\boldsymbol{\varphi}^1} \log_e\left(\frac{\boldsymbol{\varphi}^0}{q}\right),$$

which has derivatives with respect to the parameters

$$\frac{\partial p}{\partial \boldsymbol{\varphi}^0} = \frac{1}{\boldsymbol{\varphi}^0\, \boldsymbol{\varphi}^1},$$

$$\frac{\partial p}{\partial \boldsymbol{\varphi}^1} = \frac{-p\left(\mathbf{q},\boldsymbol{\varphi}\right)}{\boldsymbol{\varphi}^1}.$$

Finally, the derivative of the inverse demand function of the power law with respect to demand is

$$\frac{\partial p}{\partial q} = \frac{-1}{\boldsymbol{\varphi}^1\, q}.$$

We note from these formulae for the power law and the exponential models of demand that the parameter $\boldsymbol{\varphi}^0$ has the same role in both models of scaling the base level of demand whilst that of the parameter $\boldsymbol{\varphi}^1$ differs between the two models.

These examples of demand functions are presented to show how they can be manipulated readily to provide the inverse demand function and derivatives that will be required for the sensitivity analysis that follows below. Other forms of demand function could equally be adopted, according to preferences and needs.

*2.3 The supply model*

In the context of supply-demand equilibrium for travel through a network, the supply side is represented by the operation of the network. In the present case, this is taken as the conflation of the link (and any other) cost functions and the equilibration process within the network that together determine the relationship between the vector amount **q** of travel that takes place between the origin-destination pairs and the vector **G** of costs of travel. In a corresponding manner to the demand functions, we suppose that this relationship is influenced by some vector $\boldsymbol{\psi}$ of parameters. Thus we represent the supply model in the vector form $\mathbf{G}(\mathbf{q}, \psi)$. Variations in the supply side, including the network equilibration process, can be represented through variations in the vector of parameters $\boldsymbol{\psi}$.

By way of example, we consider the Bureau of Public Roads (BPR) polynomial link cost functions. Thus the cost $c_a$ of travel associated with link $a$ when the flow on that link is $v_a$ is given by

$$c_a\left(v_a, \boldsymbol{\psi}\right) = \psi_a^0 + \psi_a^1 \left(\frac{v_a}{\psi_a^3}\right)^{\psi_a^2} .$$

The parameter $\psi_a^0$ corresponds to the free-flow travel time on link $a$, whilst the parameter $\psi_a^3$ is often interpreted as link capacity, though there is no requirement that flow remain below this value. When $v_a = \psi_a^3$, the travel time is equal to $\psi_a^0 + \psi_a^1$, and the ratio $\psi_a^1 / \psi_a^0 = 0.15$ is often used. The exponent parameter $\psi_a^2$ is often set to 4, giving rise to a fourth power relationship between cost and flow.

The derivative of this cost flow relationship with respect to flow is given as

$$\frac{\partial c_a}{\partial v_a} = \frac{\psi_a^2\ \psi_a^1}{\psi_a^3}\left(\frac{v_a}{\psi_a^3}\right)^{\left(\psi_a^2-1\right)} = \frac{\psi_a^2}{v_a}\left(c_a\left(v_a, \boldsymbol{\psi}\right) - \psi_a^0\right).$$

Derivatives with respect to each of the parameters can be found readily for this relationship, but the one that is of greatest interest is that with respect to the capacity parameter $\psi_a^3$. Thus

$$\frac{\partial c_a}{\partial \psi_a^3} = \frac{-\psi_a^2\ \psi_a^1}{\psi_a^3}\left(\frac{v_a}{\psi_a^3}\right)^{\psi_a^2} = \frac{\psi_a^2}{\psi_a^3}\left(c_a\left(v_a, \boldsymbol{\psi}\right) - \psi_a^0\right)$$

*2.4 Analysis of network equilibrium*

We now consider the way in which an equilibrium assignment of traffic to the network is affected by variations in certain quantities. We consider first variations in the amount of travel between origin-destination pairs, and then consider variations in the parameters of the link cost functions. The analysis that is presented here is generic and can be applied to any form of link cost functions and variations in any of their parameters. In the context of the equilibrium between supply and demand, all of this analysis refers to the supply side.

*2.5 Demand-supply equilibrium*

We are now in a position to express explicitly and in compact form a result for the sensitivity of the demand-supply equilibrium point with respect to variations in the parameters $\varphi$ of the demand functions and $\psi$ of the link cost functions. These results draw upon the analysis of network equilibrium developed above. The condition for supply-demand equilibrium is that the cost of travel should be equal to the inverse of the demand function. Thus for demand-supply equilibrium, we require $\mathbf{p}(\mathbf{q}, \varphi) = \mathbf{G}^*(\mathbf{q}, \psi)$, where $\mathbf{G}$ is the vector of route costs. From this, we have

$$\mathbf{p}(\mathbf{q}, \varphi) - \mathbf{G}^*(\mathbf{q}, \psi) = \mathbf{0} .$$

This has the form of an implicit function for the origin-destination flows $\mathbf{q}$ which depend on the parameters $\varphi$ and $\psi$. Proceeding accordingly, we note that if the system is to remain in equilibrium, then the total derivative of the left-hand side of this expression with respect to the parameters $\varphi$ is zero. Thus denoting $\mathbf{q}^{\boldsymbol{\varphi}}$ as the matrix of derivatives of the equilibrium origin-destination flows with respect to the parameters $\varphi$ of the demand functions, with elements $q_{od,i}^{\boldsymbol{\varphi}} = \mathrm{d}\, q_{od} / \mathrm{d}\, \boldsymbol{\varphi}_i$ , we have $\mathbf{P}^{\boldsymbol{\varphi}} + \left(\mathbf{P} - \mathbf{K}\right)\mathbf{q}^{\boldsymbol{\varphi}} = 0$ which can be rearranged to give the expression

$$\mathbf{q}^{\boldsymbol{\varphi}} = -\left(\mathbf{P} - \mathbf{K}\right)^{-1} \mathbf{P}^{\boldsymbol{\varphi}} .$$

Similarly, denoting $\mathbf{q}^{\psi}$ as the matrix of derivatives of the equilibrium origin-destination flows with respect to the parameters $\psi$ of the link cost functions, with elements $q_{od,i}^{\psi} = \mathrm{d}\, q_{od} / \mathrm{d}\, \psi_i$ , we have $-\mathbf{K}^{\psi} + \left(\mathbf{P} - \mathbf{K}\right)\mathbf{q}^{\psi} = 0$ which can be rearranged to give the expression

$$\mathbf{q}^{\psi} = \left(\mathbf{P} - \mathbf{K}\right)^{-1} \mathbf{K}^{\psi} .$$

These two expressions for $\mathbf{q}^{\boldsymbol{\varphi}}$ and $\mathbf{q}^{\psi}$ provide a means to estimate the response of the demand-supply equilibrium to changes in either or both of the sets of parameters $\varphi$ of the demand functions and $\psi$ of the link cost functions. These expressions in their turn make use of the sensitivity analysis of the equilibrium assignment of the origin-destination flows to routes through the network.

*2.6 Linear sensitivity analysis*

We now consider the application of sensitivity analysis to estimate the effect of a deviation in the values of parameter from ones at which the performance of the transport system is known. Consider some generic measure of performance as $\theta$ , and suppose that it depends in some way on the vectors of parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ . We denote as $\boldsymbol{\varphi}^0$ and $\boldsymbol{\psi}^0$ the reference values of these vectors at which the performance of the system is known, as $\boldsymbol{\theta}_{\varphi}$ and $\boldsymbol{\theta}_{\psi}$ respectively the row vectors of first derivatives of $\theta$ with respect to the parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, and subsequent derivatives similarly. Let $\theta_{\boldsymbol{\varphi}}\left(\boldsymbol{\varphi}^0, \boldsymbol{\psi}^0\right)$ and $\theta_{\psi}\left(\boldsymbol{\varphi}^0, \boldsymbol{\psi}^0\right)$ be the values of the derivatives of $\theta$ with respect to $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ respectively, evaluated at the point $\left(\boldsymbol{\varphi}^0, \boldsymbol{\psi}^0\right)$. Then using a second order power series expansion around the reference values, the reliability measure can be approximated as:

$$\theta(\boldsymbol{\varphi},\boldsymbol{\psi}) \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\varphi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\psi}}\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right) + \\ + \tfrac{1}{2}\left[\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\varphi\varphi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + 2\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\varphi\psi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + \left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\psi\psi}}\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)\right].$$

The sensitivity analysis of supply-demand and network equilibrium presented in the Appendix can be used to furnish the first derivatives $\boldsymbol{\theta}_{\boldsymbol{\varphi}}$ and $\boldsymbol{\theta}_{\boldsymbol{\psi}}$. However, further analysis of the equilibrium process would be required to establish expressions for the second order derivatives. If the first derivatives alone are used to estimate the value of $\theta$ then the sensitivity analysis will be linear. The resulting linear approximation would then be given by

$$\theta(\boldsymbol{\varphi},\boldsymbol{\psi}) \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\varphi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\psi}}\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right).$$

We see directly from this that any effect due to interactions between the supply and the demand variations will be of second order in their magnitude.

If the parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have statistical distribution with means $\bar{\boldsymbol{\varphi}}$ and $\bar{\boldsymbol{\psi}}$ respectively, then the linear approximation for the expected value of $\theta$ is

$$\mathrm{E}\left[\theta(\boldsymbol{\varphi},\boldsymbol{\psi})\right] \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\varphi}}\left(\bar{\boldsymbol{\varphi}}-\boldsymbol{\varphi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\psi}}\left(\bar{\boldsymbol{\psi}}-\boldsymbol{\psi}^0\right).$$

In the case that the reference values of the parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are equal to their means so that $\boldsymbol{\varphi}^0 = \bar{\boldsymbol{\varphi}}$ and $\boldsymbol{\psi}^0 = \bar{\boldsymbol{\psi}}$, this linear approximation reduces to

$$\mathrm{E}\left[\theta(\boldsymbol{\varphi},\boldsymbol{\psi})\right] \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right).$$

However, in the context of network reliability it is usually more convenient to consider values of parameters that represent normal operation as reference values, even though this might well result in loss of accuracy by comparison with use of the mean because of differences between the values of the derivatives $\boldsymbol{\theta}_{\boldsymbol{\varphi}}$ and $\boldsymbol{\theta}_{\boldsymbol{\psi}}$.

Returning to the second order approximation for the measure of network performance, the second order estimate of the expected value of the performance measure $\theta$ is given as

$$\mathrm{E}\left[\theta(\boldsymbol{\varphi},\boldsymbol{\psi})\right] \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\varphi}}\left(\bar{\boldsymbol{\varphi}}-\boldsymbol{\varphi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\psi}}\left(\bar{\boldsymbol{\psi}}-\boldsymbol{\psi}^0\right) + \\ + \tfrac{1}{2}\mathrm{E}\left[\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\varphi\varphi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + 2\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\varphi\psi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + \left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\psi\psi}}\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)\right].$$

In the case that the perturbations in the values of the parameters $\boldsymbol{\varphi}$ and those in $\boldsymbol{\psi}$ are mutually independent, the covariance term involving the mixed second-order derivatives vanishes in expectation, though the separate variance terms remain, so that this reduces to

$$\mathrm{E}\left[\theta(\boldsymbol{\varphi},\boldsymbol{\psi})\right] \approx \theta\left(\boldsymbol{\varphi}^0,\boldsymbol{\psi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\varphi}}\left(\bar{\boldsymbol{\varphi}}-\boldsymbol{\varphi}^0\right) + \boldsymbol{\theta}_{\boldsymbol{\psi}}\left(\bar{\boldsymbol{\psi}}-\boldsymbol{\psi}^0\right) + \\ + \tfrac{1}{2}\mathrm{E}\left[\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\varphi\varphi}}\left(\boldsymbol{\varphi}-\boldsymbol{\varphi}^0\right) + \left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)'\boldsymbol{\theta}_{\boldsymbol{\psi\psi}}\left(\boldsymbol{\psi}-\boldsymbol{\psi}^0\right)\right].$$

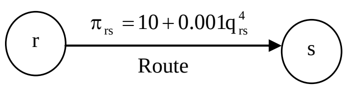
This expression depends on the variance of each of the supply and demand, weighted by the corresponding second derivatives $\theta_{\varphi\varphi}$ and $\theta_{\psi\psi}$.

From this analysis, we conclude that any effect on the expected value of the performance measure that is due to interactions between the supply and the demand variations will be of third order or smaller in their magnitude.

In summary, a first-order sensitivity analysis can be used to estimate the effect on a performance measure of variations in the values of parameters that characterize supply and demand. Because it is first-order, the resulting estimate will not depend on interactions between variations in supply and demand. This analysis can be used to estimate the expected value of the performance measure which includes a correction to the value of the performance measure at a reference combination of parameter values which is linear in the deviation of these from their respective means. A second-order sensitivity analysis could be developed to estimate these values more accurately. However, consideration of its use shows that if the variations in the parameters that characterize the supply are independent of those for the demand, then a second order estimate of the expected value of a performance measure based upon it will not be influenced by interactions between variations in supply and demand.

## 3. ILLUSTRATIVE EXAMPLE FOR THE ISSUES OF DSR

Figure 1 shows a simple one-route (or one-link) network that is used to illustrate the principal issues of DSR, which form the basis of the DSR formulation. In addition, further assumptions are adopted to facilitate the presentation of the essential ideas without loss of accuracy or leading to an erroneous conclusion.




Demand Function Before LDE Occurs
$q_{rs} = -\pi_{rs}/3 + 10$ .
The Value of Positive Item of the RTD When LDE Occurs
$\psi_{rs} = 5$ .
Demand Function after Shift of Demand
$q_{rs} = -\pi_{rs}/3 + 10 + \psi_{rs}$
$= -\pi_{rs}/3 + 15$

FIGURE 1: One-route network

In this network, link travel time $\pi_{rs}$ is assumed as a quadratic function of link flow such as the one adopted by BPR (Bureau of Public Road). The free-flow travel time on the link is given with 10 minutes (min). Figure 2 illustrates the relationship between travel demand and network supply. The RTD function is in a linear form of $q_{rs} = -\pi_{rs}/3 + 10 + \psi_{rs}$, where $\psi_{rs}$ is the positive item related to the factors of the LDE, for example, the explanatory variables of holiday frequency and personnel income for recreational travel (see Yai et al., 1995). The item $\psi_{rs}$ is 0 when there is no LDE, of which $\pi_{rs} = -3q_{rs} + 30$ is the inverse linear function as shown in Figure 2. Suppose that the occurrence of LDE is at a probability of 50%. As the LDE occurs, the item $\psi_{rs}$ equals to 5 units, the travel demand function would then be shifted to $q_{rs} = -\pi_{rs}/3 + 15$ , at which $\pi_{rs} = -3q_{rs} + 45$ is another inverse linear function as displayed in Figure 2.

There are dual problems to consider how a network satisfies the increment travel demand or how a degradable network incurs the decrement travel demand. The effect of the degradable network on OD flows will be discussed later in this paper. The following concentrates on the aspects for assessing the impacts of the LDE (i.e. the increment travel demand due to LDE).

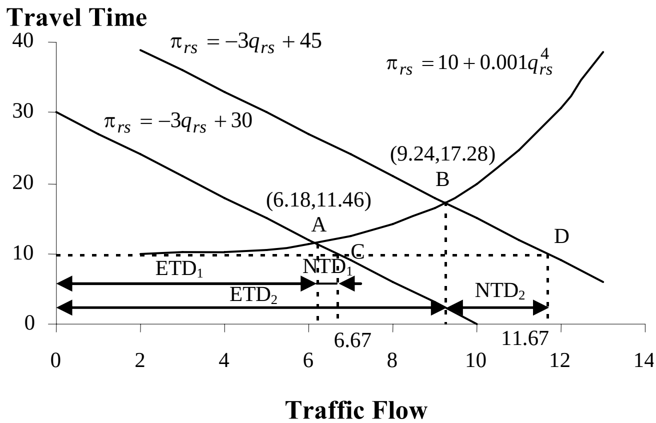



FIGURE 2: Equilibrium of travel demand and network supply

It can be noticed from Figure 2 that, if the network can provide a service level at the free-flow travel time condition (i.e. 10 min) for travel, the latent travel demand $LTD_1$ will be equal to 6.67 at the point C before the occurrence of LDE. If the network cannot provide the free-flow service level, some OD trips will not be satisfied or revealed due to the slight congestion of network before the LDE occurs. The non-revealed travel demands are $NTD_1$=6.67-6.18=0.49. The equilibrium travel demands attain $ETD_1$=6.18. The latent travel demands are then $LTD_1$=6.67. There are $DSI_1$=$ETD_1$/ $LTD_1$ = 92.65% of latent travel demands that can be revealed.

When the LDE occurs, the equilibrium travel demand attains $ETD_2$=9.24, but the non-revealed travel demand also increases to $NTD_2$=11.67-9.24=2.43. It means that the latent travel demand is $LTD_2$ = 11.67 at the point D. Only $DSI_2$=$ETD_2$/ $LTD_2$ = 79.18% of latent travel demands that can be revealed.

The DSR is 100% with the DSI in the region of [0, 79.18%]. It falls to 50% while the DSI locates in the region of [79.18%, 92.65%]. Then it becomes zero as the DSI is greater than 92.65%. The results of the DSR are shown in Figure 3. When the specified DSI is less than 79.18%, the DSR is 100%. It means that once the road users accept the lowest DSI, the network would satisfy their needs completely. If the specified DSI is greater than 92.65%, the DSR becomes 0%. It implies that when the need of road users exceeds the maximal ability of network, the probability of the network satisfying the latent travel demand will approach to 0%. If the specified DSI is in the range between 79.18% and 92.65%, then the DSR is 50%.

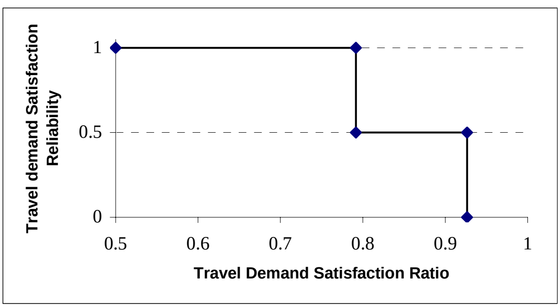



FIGURE 3: Travel demand satisfaction reliability

It is worthwhile to consider the travel time, system and OD sub-system reliabilities in this example. Figure 2 shows that the ETD and equilibrium travel time are located at the point A (6.18, 11.46) when none of the LDE occurs. The ETD and equilibrium travel time would move to the point B (9.24, 17.28) while the LDE occurs. Both the points A and B are at the non-degraded network state. It can be argued that whether point A or B is the denominators for measuring the indicators of network service or the decrement rate of OD flow as defined in Table 1. However, if the state of LDE is fixed at either point A or B, the network service level would then be equal to 1 and the decrement rate of OD flows equals 0. The reliabilities of the travel time, system and OD sub-system are thus all 100% as discussed in the above. Hence in line of the original definitions, these network reliability measures are generally **over-estimated**.

However, the definition of the unique reference point can be extended to the state of non-degradable network and to the situation without LDE. Point A is thus chosen as the unique reference point. The decrement rate of OD flow is (11.46-17.28)/11.46 = -57% while the LDE occurs. The negative rate of OD flow is unreasonable in this case. The network service level equals to 17.28/11.46=1.51 by modifying the original definitions. The occurrence of LDE causes the network service level to attain at 1.51 with a probability of 50%. It implies that the travel time reliability is only 50% if the required travel time is not greater than 1.51×11.46=17.28 (min). The travel time reliability can indicate whether the network can or cannot satisfy the traveller’s need in corresponding to the travel time by OD pair. However, there are a wide variety of the needs in terms of travel time for each traveller. It is important to indicate how much latent travel demand can or cannot be revealed for thorough assessment of the network performance in this case.

In corresponding to the network service level of 1.51, the travel demand satisfaction ratio is 70.18%. It implies that when the LDE occurs, the demands of the travellers who have the specified network service levels more than 1.51 is 70.18% of the LTDs. In other words, there is a probability of 50% that travellers would have travel time less than 1.51×11.46=17.28 (min) and the DSI is more than 70.18%.

On the other hand, the capacity reliability is also illustrated here. With reference to the network shown in Figure 1, it is assumed that the referenced OD demand is 6.18 and the link capacity is 7. Then, the reserve capacity is 7/6.18=1.133. From the viewpoint of capacity reliability, the network can only satisfy the demand level less than 1.133. It implies that only 6.18 OD trips can be revealed. However, in view of the function of the RTD, i.e. $\pi_{rs} = -3q_{rs} + 45$, the network can cater for the travel demand of 7 and queues with extra queue time of $[(45-3\times7)-(10+0.001\times7^4)]=11.599$ (min) in this case. In other words, the demand levels that the network can accommodate are obviously **under-estimated** by the reserve capacity as no queues are allowed. It also implies that the capacity reliability is always **under-estimated**, particularly when queues are always inevitable during the peak periods in practice.

## 4. DEFINITION AND FEATURES OF DSR

Before the DSR is defined in detail together with the mathematical formulation, some preliminary concepts of network reliability are briefly reviewed in the following. In general, the benchmark and performance indicator should be considered respectively in assessing the network reliability for some specific purposes. In addition, the elastic demand function must be taken into account when assessing the DSR. These will help provide deeper insights to the definition of DSR.

*4.1 Benchmark for assessing network reliability*

Assessment of network reliability has to be scaled in conformation to a frame of reference so as to provide certain physical significance and specified standard. The original point with proper scale and direction (increasing or decreasing the reliability) based on the reference frame can be indicated as a benchmark.

In the model proposed by Iida and Wakayabashi (1989) for assessing terminal reliability of road network, one of the binary connectivity state, i.e. connect or disconnect, could be considered as its benchmark for evaluation of the connectivity reliability. The formulation of travel time reliability given by Asakura and Kashiwadani (1991) implies that its benchmark is the travel time in the state of free-flow, at which traveller can arrive to his (or her) destination with the least travel time. It can also be implied that the benchmark suggested by Asakura (1998) is the equilibrium travel time at the state of non-degradable network. The ETD at the state of non-degradable network can be equivalent to the benchmark for system and OD sub-system reliability (see Du and Nicholson, 1997). The benchmark for capacity reliability can be referred to as the reserve capacity at the state of non-degradable network with an initially given OD trip matrix, which is the maximum reserve capacity that a network can provide under the constraints of link capacities (see Chen et al., 1999 in detail). In addition, the road network cannot be unconstrained for measuring the reserve capacity. Otherwise, the reserve capacity is unreasonable to approach infinity.

In view of the previous related studies and the impacts of LDE, it is suggested that the set of non-degradable network and free-flow condition is a stable reference frame rather than the other set of combination. In this case, latent demand event can be released freely (see Dowling and Colman, 1995) and thus the LTD is estimated. Moreover, for assessing the DSR, both ETD and LTD are the key factors for determination of the RTD. DSI also indicates the level at which a road network satisfies the latent travel demand, with $\theta_S = 1$ in the optimum state of non-degradable network and free-flow conditions as the benchmark for assessing the DSR.

*4.2 Performance indicator for analysis of network reliability*

In the system engineering and management discipline, the system reliability to meet some specifications is normally adopted as performance indicators, such as time to failure, time to restore system, response time to serve customer and life cycle of development of new product and etc. Hence, the probability to keep an indicator within a certain limit can be measured accordingly. The system reliability is thus quantified.

In reference to the previous definitions of road network reliability, the performance indicators are summarised in Table 1. As discussed in the above analysis, DSI ($\theta_S$) is suggested as the performance indicator for assessing the DSR. It can reflect the ability of a network satisfying and/or revealing the latent travel demand.

*4.3 Elastic demand in formulating DSI and DSR*

As noticed from the above, DSI and DSR cannot be formulated without the estimation of LTD. The constant travel demand function is not able to identify the ETD and LTD coherently and simultaneously. The elastic travel demand function should therefore be employed for the DSR analysis. For simplicity without the loss of generality and accuracy, the OD flow is assumed to be a function of travel time by OD pair. However, the other factors such as income and population can be incorporated into the travel demand function easily but they do not have direct interaction with the route choices.

Two forms of travel demand function are usually used to predict the ETD and LTD. The linear travel demand function can be expressed as the form of $q_{rs}(\pi_{rs}) = a_0 + a_1\pi_{rs}$, where $a_0 > 0$ and $a_1 \leq 0$ are the function coefficients. If $a_1$ approaches to zero, the demand function becomes constant. When the LDE occurs with the expression of the positive item of $\psi_{rs}$, the linear demand curve will be shifted to the RTD, $q_{rs}(\pi_{rs}, \psi_{rs}) = \psi_{rs} + a_0 + a_1\pi_{rs}$. It implies that the OD flows will be increased by the additional demand $\psi_{rs}$ even if the OD travel time is remained constant. The inverse of the linear demand function is thus in the form of $\pi_{rs}(q_{rs}, \psi) = \psi_{rs}/a_1 + a_0/a_1 + q_{rs}/a_1$. The advantage of linear travel demand function is its direct proportional interpretation of the RTD while its disadvantage is the restriction of its uses to a limited condition with the assumption of linear relationship between the OD demand and travel time.

The travel demand function being commonly used by most researchers and planners is in the elastic form of $q_{rs}(\pi_{rs}) = Q_{rs}\pi_{rs}^{-\xi}$, where $Q_{rs} > 0$ is the OD flow at the travel time $\pi_{rs} = 1$. $\xi = -\dfrac{dq_{rs}}{q_{rs}} \Big/ \dfrac{d\pi_{rs}}{\pi_{rs}}$ is referred to as the *demand elasticity*. As the OD travel time increases by one percent, the OD flows decrease $\xi$ percent. If $\xi$ approaches to zero, the elastic demand function will become constant. On the basis of the elastic form, the RTD is expressed as $q_{rs}(\pi_{rs}, \text{LDE}) = Q_{rs}\pi_{rs}^{-\xi} f_{rs}(\text{LDE})$, where $f_{rs}(\text{LDE}) \geq 1$ is a functional item affecting proportionally the OD flows by occurrence of the LDE. If none of LDE occurs, then $f_{rs}(\text{LDE}) = 1$. Let $\rho_{rs} = Q_{rs}[f(\text{LDE}) - 1] \geq 0$, the RTD is rewritten as $q_{rs}(\pi_{rs}, \rho_{rs}) = (Q_{rs} + \rho_{rs})\pi_{rs}^{-\xi}$. When the LDE occurs with the value of positive item of $\rho_{rs}$, the demand curve will be shifted to $q_{rs}(\pi_{rs}, \rho_{rs}) = (Q_{rs} + \rho_{rs})\pi_{rs}^{-\xi}$. It implies that the OD flows will be increased by the additional demand $\rho_{rs}\pi_{rs}^{-\xi}$ even if the OD travel time is kept constant. The inverse of the elastic demand function is thus in the form of $\pi_{rs}(q_{rs}, \rho_{rs}) = [(Q_{rs} + \rho_{rs})/q_{rs}]^{1/\xi}$. The advantage of elastic demand function is its unchangeable elasticity, which is easily estimated and/or calibrated by the survey data. However, the OD flows may be over-estimated sometimes particularly when the OD travel time approaches to a very small value.

Either the variable $\psi_{rs}$ or $\rho_{rs}$ can be defined as random variables due to the LDE. Because the RTD is a derived function of the $\psi_{rs}$ or $\rho_{rs}$, it is also considered as random variable. The random variables can be adopted as discrete or continuous probability. For simplicity without loss of generality, only the elastic form of travel demand function and discrete random variables are used in the following discussion.

*4.4 Mathematical formulation of DSR*

As discussed above, the RTD can be simply expressed as the form of $q_{rs}(\pi_{rs},\rho_{rs})$. The inverse function of $q_{rs}(\pi_{rs},\rho_{rs})$ is defined as $\pi_{rs}(q_{rs},\rho_{rs})$. Let $q_{rs}^*$ and $\pi_{rs}^*$ be defined as the LTD and the free-flow travel time by OD pair in a non-degradable network respectively. $\rho_{rsi}$ is denoted as the *i*-th random state of the RTD for OD pair *r* and *s*. The relationship of $q_{rs}^*$ and $\pi_{rs}^*$ is

$$q_{rs}^* = q_{rs}^*(\rho_{rsi}) = q_{rs}(\pi_{rs}^*,\rho_{rsi}) \tag{1}$$

on the basis of the proposed benchmark for assessing the DSR. Note that $q_{rs}^*$ is a monotonically increasing function of $\rho_{rsi}$. The value of $\pi_{rs}(q_{rs}^*,\rho_{rsi})$ is equivalent to the free-flow travel time, i.e. $\pi_{rs}^*$.

Let $\hat{q}_{rs}$ be defined as the ETD, $\hat{q}_{rs} = q_{rs}(\hat{\pi}_{rs},\rho_{rs})$, where $\hat{\pi}_{rs}$ is the equilibrium travel time. Suppose that $\kappa_j$ is the *j*-th random state of the degradable network, which occurs with the probability $P_{\kappa j}$. The $P_{\kappa j}$ can be expressed for any types of probability distribution in agreement with the real scenario. $\hat{\pi}_{rs}$ is the result from the equilibrium of the RTD and network supply. Let $\rho$ be the vector of $\rho_{rs}$. Denoted $\rho_i$ as the *i*-th state of RTD, which occurs with probability $P_{\rho i}$. The $P_{\rho i}$ can be expressed for any types of probability distribution in conformation to the reality. Hence, the ETD at both the *i*-th random state of RTD and the *j*-th random state of degradation of the network is a function of both the random variables $\rho_{rsi}$ and $\kappa_j$. They are expressed as

$$\hat{q}_{rs} = \hat{q}_{rs}(\rho_i,\kappa_j) \text{ and } \hat{\pi}_{rs} = \hat{\pi}_{rs}(\rho_i,\kappa_j). \tag{2}$$

If the states of both the random variables $\rho_i$ and $\kappa_j$ are fixed, the coefficients of the RTD and network are deterministic. The ETD would then become the deterministic user equilibrium with elastic demand. The algorithm proposed by LeBlanc and Farhangian (1981) can be used to determine the ETD. The expression of DSI is therefore determined. That is

$$\theta_{rs} = \theta_{rs}(\rho_i,\kappa_j) = \hat{q}_{rs}(\rho_i,\kappa_j) \Big/ q_{rs}^*(\rho_{rsi}). \tag{3}$$

$\theta_{rs}$ is a function of the random variables $\rho_{rsi}$ and $\kappa_j$. Let $\theta$ be the aggregate DSI for the whole network and thus

$$\theta = \theta(\rho_i,\kappa_j) = \sum_{rs} \hat{q}_{rs}(\rho_i,\kappa_j) \Big/ \sum_{rs} q_{rs}^*(\rho_{rsi}). \tag{4}$$

$\theta_{rs}$ is used to determine the DSI of an OD pair, while $\theta$ is used to determine the DSI of the transportation network. For simplicity without the loss of accuracy to present the essential concept, the random variables $\rho_i$ and $\kappa_j$ are assumed to be independent. Suppose that $1 \geq \theta_s \geq 0$ is denoted as the specified travel demand satisfaction ratio. $R_{rs}$ is the DSR for OD pair $r \rightarrow s$ and R is the DSR for the whole network. The DSR is the probability that the DSI is not less than $\theta_s$. Hence, The DSR of an OD pair and of the transportation network can be defined as

$$R_{rs} = P(\theta_{rs}(\rho_i,\kappa_j) \ge \theta_s \big| \rho_i,\kappa_j \text{ for all } i \text{ and } j) = \sum_{(i,j)\in\{(i,j)|\theta_{rs}(\rho_i,\kappa_j)\ge\theta_s\}} P_{\rho i}P_{\kappa j} \quad , \tag{5}$$

and

$$R = P(\theta(\rho_i,\kappa_j) \ge \theta_s \big| \rho_i,\kappa_j \text{ for all } i \text{ and } j) = \sum_{(i,j)\in\{(i,j)|\theta(\rho_i,\kappa_j)\ge\theta_s\}} P_{\rho i}P_{\kappa j} \quad . \tag{6}$$

The averages of the DSI are calculated as

$$\bar{\theta}_{rs} = \sum_{i,j} \theta_{rs}(\rho_i,\kappa_j)P_{\rho i}P_{\kappa j} \text{ and } \bar{\theta} = \sum_{i,j}\theta(\rho_i,\kappa_j)P_{\rho i}P_{\kappa j} \, . \tag{7}$$

*4.5 Some features of DSI and DSR*

In reference to the reliability definitions given in Table 1, it is shown as follows that the proposed DSR can be equivalent to the travel time reliabilities (I) and (II) as well as the OD sub-system reliability under certain conditions. From the inverse of the demand function, the equilibrium travel time is identical with

$$\hat{\pi}_{rs} = \pi_{rs}(\hat{q}_{rs},\rho_{rs}) \, . \tag{8}$$

As noted from the above, the travel demand function is adopted as an elastic form:

$$\pi_{rs}(\theta q_{rs},\rho_{rs}) = [(Q_{rs}+\rho_{rs})/(\theta q_{rs})]^{1/\xi} = \frac{1}{\theta^{1/\xi}}\pi_{rs}(q_{rs},\rho_{rs}) \, . \tag{9}$$

In general the travel demand function, i.e. equation (9), is a strictly decreasing function of the OD flow $q_{rs}$ and a strictly increasing function of the LDE item $\rho_{rs}$. In the following analysis, the subscripts of the random state of both RTD and degradable network are ignored.

*4.6 Travel time reliability (I)*

As shown in Table 1, the travel time reliability has been proposed as one of the measures on the *travel time* by OD pair. Note that the travel time reliability (I) does not include the random degradation of the network. It can be deduced from equation (5) that

$$\begin{aligned} P\left(\theta_{rs}(\rho_{rs},0) \ge \theta_s\right) &= P\left(\frac{\hat{q}_{rs}(\rho,0)}{q^*_{rs}(\rho_{rs})} \ge \theta_s\right) \\ &= P\left(\hat{q}_{rs}(\rho,0) \ge \theta_s q^*_{rs}(\rho_{rs})\right) \\ &= P\left(\pi_{rs}(\hat{q}_{rs}(\rho,0),\rho_{rs}) \le \pi_{rs}(\theta_s q^*_{rs}(\rho_{rs}),\rho_{rs})\right) \\ &= P\left((\hat{\pi}_{rs}(\rho,0) \le \pi_{rs}(q^*_{rs}(\rho_{rs}),\rho_{rs})/\theta_s^{1/\xi}\right) \\ &= P\left(\hat{\pi}_{rs}(\rho,0) \le \pi^*_{rs}/\theta_s^{1/\xi}\right). \end{aligned} \tag{10}$$

Let $\theta_T$ be $\pi^*_{rs}/\theta^{1/\xi}$, equation (10) is equivalent to the travel time reliability (I).

*4.7 Travel time reliability (II)*

If the travel time reliability (II) does not consider the effects of the LDE as originally assumed, it can then be deduced from equation (5) that

$$\begin{aligned}
P\left(\theta_{rs}(0,\kappa) \ge \theta_s\right) &= P\left(\frac{\hat{q}_{rs}(0,\kappa)}{q^*_{rs}} \ge \theta_s\right) \\
&= P\left(\hat{q}_{rs}(0,\kappa) \ge \theta_s q^*_{rs}\right) \\
&= P\left(\pi_{rs}(\hat{q}_{rs}(0,\kappa),0) \le \pi_{rs}(\theta_s q^*_{rs},0)\right) \\
&= P\left(\hat{\pi}_{rs}(0,\kappa) \le \pi_{rs}(q^*_{rs},0) \big/ \theta_s^{1/\xi}\right) \\
&= P\left(\hat{\pi}_{rs}(0,\kappa)/\hat{\pi}_{rs}(0,0) \le \pi^*_{rs} \big/ \hat{\pi}_{rs}(0,0) \big/ \theta_s^{1/\xi}\right),
\end{aligned} \tag{11}$$

where $\hat{\pi}_{rs}(0,0)$ is the equilibrium travel time at the non-degraded state of the network. Let $\theta_L$ be $\pi^*_{rs} \big/ \hat{\pi}_{rs}(0,0) \big/ \theta_s^{1/\xi}$. It should be pointed out that equation (11) is consistent with the travel time reliability (II) as defined by Asakura (1998).

If the travel time reliability (II) is extended to include the impacts of LDE as discussed above, it can be deduced from equation (5) that

$$\begin{aligned}
P\left(\theta_{rs}(\rho_{rs},\kappa) \ge \theta_s\right) &= P\left(\frac{\hat{q}_{rs}(\rho,\kappa)}{q^*_{rs}(\rho_{rs})} \ge \theta_s\right) \\
&= P\left(\hat{q}_{rs}(\rho,\kappa) \ge \theta_s q^*_{rs}(\rho_{rs})\right) \\
&= P\left(\pi_{rs}(\hat{q}_{rs}(\rho,\kappa),\rho_{rs}) \le \pi_{rs}(\theta_s q^*_{rs}(\rho_{rs}),\rho_{rs})\right) \\
&= P\left(\hat{\pi}_{rs}(\rho,\kappa) \le \pi_{rs}(q^*_{rs}(\rho_{rs}),\rho_{rs}) \big/ \theta_s^{1/\xi}\right) \\
&= P\left(\hat{\pi}_{rs}(\rho,\kappa)/\hat{\pi}_{rs}(0,0) \le \pi^*_{rs} \big/ \hat{\pi}_{rs}(0,0) \big/ \theta_s^{1/\xi}\right).
\end{aligned} \tag{12}$$

Let $\theta_L$ be $\pi^*_{rs} \big/ \pi_{rs}(0,0) \big/ \theta_s^{1/\xi}$. It can easily be seen that equation (12) is consistent with the travel time reliability (II) in this case.

*4.8 System and OD sub-system reliability*

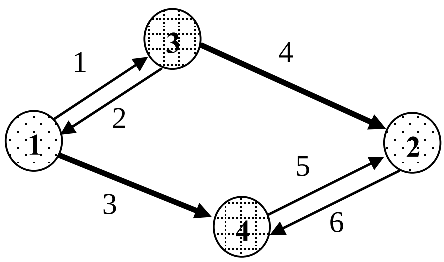
If the OD sub-system reliability does not include the random shift of the demand, it can be deduced from equation (5) that

$$\begin{aligned}
P\left(\theta_{rs}(0,\kappa) \ge \theta_s\right) &= P\left(\frac{\hat{q}_{rs}(0,\kappa)}{q^*_{rs}} \ge \theta_s\right) \\
&= P\left(\frac{\hat{q}_{rs}(0,0) - \hat{q}_{rs}(0,\kappa)}{\hat{q}_{rs}(0,0)} \le \frac{\hat{q}_{rs}(0,0) - \theta_s q^*_{rs}}{\hat{q}_{rs}(0,0)}\right),
\end{aligned} \tag{13}$$

where $\hat{q}_{rs}(0,0)$ is the ETD at the non-degraded state of the network. Let $\theta_O$ be $(\hat{q}_{rs}(0,0) - \theta_s q^*_{rs}) / \hat{q}_{rs}(0,0)$. Therefore, equation (13) is equivalent to the OD sub-system reliability as proposed by Du and Nicholson (1997).

## 5. NUMERICAL EXAMPLES

Figure 4 shows an example network for illustration of the relationships between the proposed performance indicator DSR and the other existing reliability measures. This example network, which consists of 6 one-way links, 4 nodes and two OD pairs, is used to discuss the properties of the DSR. The link travel time functions are also given in the Figure 4. The two OD pairs are connected by nodes (1→2) and (3→4). Due to the symmetry of the example network, the OD (1→2) demand will be allocated equally between routes (or paths) 1→3→2 and 1→4→2. Similarly, the OD (3→4) demand will also be split equally between routes 3→1→4 and 3→2→4.




Link Time Function (time in minute):

$t_1(v_1) = 20 + 10(v_1/C_1)^4$ , $t_2(v_2) = 20 + 10(v_2/C_2)^4$

$t_3(v_3) = 60 + 5(v_3/C_3)^4$ , $t_4(v_4) = 60 + 5(v_4/C_4)^4$

$t_5(v_5) = 20 + 10(v_5/C_5)^4$ , $t_6(v_6) = 20 + 10(v_6/C_6)^4$

Non-degradable Capacity:

$C_1 = C_2 = C_5 = C_6 = 10$ (veh/min)

$C_3 = C_4 = 20$ (veh/min)

FIGURE 4: Example network for travel demand satisfaction reliability

The RTD functions are

$$q_{12} = (900 + \rho_{12})\pi_{12}^{-1.0} \text{ (veh/min) and } q_{34} = (900 + \rho_{34})\pi_{34}^{-1.0} \text{ (veh/min)} \qquad (14)$$

where $\rho_{12}$ and $\rho_{34}$ are the positive random variables of the LDE by OD pair. The states of the RTD and the degradable network are presented in Table 2. The variations of the RTD and the degradable network are independently stochastic. The random state $\kappa_j$ of the degradable network is denoted as $\kappa_j = [C_3, C_4]_j$ and the random state $\rho_i$ of the RTD is denoted as $\rho_i = [\rho_{12}, \rho_{34}]_i$ in this example.

In order to illustrate the essential ideas of the DSR, three cases are considered in this numerical example as follows. Case (1) - the LDE for OD (1→2) and (3→4) are assumed to occur at the same time. Case (2) - it is assumed that both the links 3 and 4 are degraded at the same time. Case (3) - combination of cases (1) and (2), i.e. both the LDE and network degradation are supposed to occur simultaneously.

Due to the symmetry of the example network, the reliability analysis results for all OD pairs are symmetrical and equal to each other. Therefore, only the results for one OD pair are shown to facilitate the presentation of the essential ideas. For simplicity, in this example, the RTD and degradable network are assumed with uniform distribution. Consequently, the probability at each of the combined random states is equal to

0.25×0.25=0.0625 for the combination of the RTD and degradable network as shown in Table 2.

TABLE 2: The parameters of RTD function and network degradation (veh/min)

| | | | | |
|---|---|---|---|---|
| Probability of latent demand event | 0.25 | 0.25 | 0.25 | 0.25 |
| RTD function parameters: $\rho_{12}$ and $\rho_{34}$ [a] (Ratio of $\rho_{rs}$ and 900) | 0 (0%) | 300 (33%) | 600 (66%) | 900 (100%) |
| Probability of degraded network | 0.25 | 0.25 | 0.25 | 0.25 |
| Link capacity: $C_3$ and $C_4$ [b] (Ratio of degraded and non-degraded capacities) | 20 (100% or non-degraded) | 15 (75%) | 10 (50%) | 5 (25%) |

Note: [a] $\rho_{12}$ and $\rho_{34}$ (Ratio of $\rho_{rs}$ and 900) are referred to the parameters in equation (14)
[b] $C_3$ and $C_4$ (Ratio of degraded and non-degraded capacities) are referred to the link capacities in Figure 4

In Case (1), both $\rho_{12}$ and $\rho_{34}$ varies randomly with the absolute values of 0, 300, 600 and 900 or the relative value ($\rho_{12}$/900 and $\rho_{34}$/900) of 0%, 33%, 66% and 100% respectively. In Case (2), when the state is non-degradable network, it means that both $C_3$ =20 and $C_4$ =20 with a probability of 25% (i.e. 0.25). Subsequently, both the $C_3$ and $C_4$ vary randomly with the absolute values of 15, 10 and 5 (veh/min) or the relative value ($C_3$ /20 and $C_4$ /20) of 75%, 50% and 25% respectively. In Case (3), the combined states of the RTD and degradable network are enumerated. The number of states is 4×4=16. Each state occurs with the chance of 0.25×0.25=0.0625.

The free-flow travel time for the two OD pairs is 80 (min) and the resultant LTDs are presented in Table 3. The latter are calculated at different random states of RTD as shown in Table 2 by using equation (14). The LTD is increasing from 11.25 to 22.50 (veh/min) as the RTD rises from the original demands to its two-fold demands. It should be noted that prediction of the LTD is impossible without using the elastic demand function.

TABLE 3: The latent travel demand of each OD pairs (veh/min)

| Free-flow travel time | RTD function parameters [a] | | | |
|---|---|---|---|---|
| | 0 (0%) | 300 (33%) | 600(66%) | 900(100%) |
| $\pi_{34}^{*} = 80$ | 11.25 [b] | 15.00 | 18.75 | 22.50 |

Notes: [a] $\rho_{12}$ and $\rho_{34}$ (Ratio of $\rho_{rs}$ and 900) are referred to the parameters in equation (14)
[b] The latent travel demand is equal to 11.25 veh/min when $\rho_{12}$ and $\rho_{34}$ equals 0 under free-flow travel condition

Using the user equilibrium assignment with elastic demand, the ETD and equilibrium travel time are determined for each of the combined states of the RTD and the degraded network as given in Table 2. The results are shown in Table 4. Note that the ETD is the equilibrium travel demand. The ETD is decreasing while its travel time is increasing when the network is degraded with reduction in capacities. However, both the ETD and equilibrium travel time are increasing with $\rho_{12}$ and $\rho_{34}$.

Let $\tilde{q}_{rs}(\rho,\kappa)$ be denoted as the total NTD at the state of $(\rho,\kappa)$, which equal to

$$\tilde{q}_{rs}(\rho,\kappa) = q_{rs}^{*}(\rho_{rs}) - \hat{q}_{rs}(\rho,\kappa) .$$

TABLE 4: The ETD (veh/min) and equilibrium travel time (min) of each OD pair

| Network degradation state | RTD function parameters [a] | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0 (0%) | | 300 (33%) | | 600 (66%) | | 900 (100%) | |
| | ETD | ETT | ETD | ETT | ETD | ETT | ETD | ETT |
| non-degraded | 11.12 | 80.95 | 14.50 | 82.76 | 17.48 | 85.83 | 20.00 | 90.00 |
| 15 (75%)[b] | 10.99 | 81.90 | 14.10 | 85.13 | 16.66 | 90.02 | 18.74 | 96.04 |
| 10 (50%) | 10.43 | 86.29 | 12.76 | 94.07 | 14.50 | 103.47 | 15.85 | 113.55 |
| 5 (25%) | 7.99 | 112.68 | 9.02 | 133.09 | 9.78 | 153.41 | 10.38 | 173.35 |

Note: ETD = Equilibrium travel demand by OD pair (veh/min)
ETT = Equilibrium travel time by OD pair (min)
[a] $\rho_{12}$ and $\rho_{34}$ (Ratio of $\rho_{rs}$ and 900) are referred to the parameters in equation (14)
[b] Link capacity, $C_3$ and $C_4$ = 15 (the ratio of degraded and non-degraded capacities = 75%)

When the LDE and network degradation occur simultaneously, the NTD is increased due to the congestion or capacity constraint of network. The increment of the NTD is

$$\Delta\tilde{q}_{rs}^{T} = \tilde{q}_{rs}(\rho,\kappa) - \tilde{q}_{rs}(0,0),$$

and is referred to as the *total loss*. It reflects the incremental loss of the latent travel demand resulted from the LDE and network degradation. In terms of the increment of the NTD, the total loss consists of three components as below. Firstly, the increment of the NTD due to the single occurrence of the LDE:

$$\Delta\tilde{q}_{rs}^{S} = \tilde{q}_{rs}(\rho,0) - \tilde{q}_{rs}(0,0).$$

Secondly, the increment of the NTD due to the single occurrence of network degradation:

$$\Delta\tilde{q}_{rs}^{D} = \tilde{q}_{rs}(0,\kappa) - \tilde{q}_{rs}(0,0).$$

Thirdly, the residual increment of the NTD (so-called the *cross loss* of the latent travel demand) when both the LDE and the degraded network:

$$\Delta\tilde{q}_{rs}^{C} = \Delta\tilde{q}_{rs}^{T} - \Delta\tilde{q}_{rs}^{S} - \Delta\tilde{q}_{rs}^{C} = \tilde{q}_{rs} - \tilde{q}_{rs}(\rho,0) - \tilde{q}_{rs}(0,\kappa) + \tilde{q}_{rs}(0,0).$$

On the basis of Tables 3 and 4, the total and cross losses of the latent travel demand are determined and summarised in Table 5. The total and cross losses are increasing with the increasing RTD and the decreasing capacity (degraded network). Furthermore, when the $\rho_{rs}$ increases to 33% and the degraded capacity decreases to 75%, the cross loss is 30% of the total loss. At the extreme state (RTD state of 100% and degraded network state of 25%), the cross loss is 53.55% of the total loss. It implies that the interaction between the LDE and degraded network causes cross effects on the evaluation of the DSR significantly.

If the cross loss did not exist, the ETD could be $\hat{q}_{rs}(\rho,\kappa) + \Delta\tilde{q}_{rs}^{C}(\rho,\kappa)$. The DSI would be $[\hat{q}_{rs}(\rho,\kappa) + \Delta\tilde{q}_{rs}^{C}(\rho,\kappa)] / q_{rs}^{*}(\rho_{rs})$. The resultant DSIs are shown in Table 6. At the state of RTD of 100% and degraded network of 25%, the difference of the DSI between the total and no-cross loss is 74.98%-46.13%= 28.85%. If the cross loss is not included for analysis of the network reliability, the DSI will be over-estimated. The cross effect has not yet been considered in the previous related studies. The reliabilities were estimated in relation to a given travel demand function (or a fixed OD trip matrix). The reliabilities were in general over-estimated.

When the degradation of network is fixed as shown in Table 6, the resultant DSR is determined and displayed in Figure 5 with reference to the network degradation states of 100% (non-degraded), 75%, 50% and 25% respectively. The larger the extent of degradation, the less is the DSR. In the network degradation state of 25%, the DSR will be 0% and 100% as the specified DSI is more than 71% and less than 46% respectively. For the network state of 75% degradation, the DSR will remain at 100% as the specified DSI is less than 83%. The DSR will be maintained at 0% until the specified DSI is less than 97%.

TABLE 5: The increment of the NTD (veh/min) from the LDE and degraded network

| Network degradation state | RTD function parameters [a] | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0 (0%) | | 300 (33%) | | 600 (66%) | | 900 (100%) | |
| | Increment of NTD | | Increment of NTD | | Increment of NTD | | Increment of NTD | |
| | Total | Cross | Total | Cross | Total | Cross | Total | Cross |
| Non-degraded | 0 [b] | 0 [c] ( 0 [d]) | 0.37 | 0 ( 0 ) | 1.14 | 0 ( 0 ) | 2.37 | 0 ( 0 ) |
| 15 (75%) [e] | 0.13 | 0 ( 0 ) | 0.67 | 0.27 (30.00) | 1.96 | 0.69 (33.01) | 3.63 | 1.13 (30.05) |
| 10 (50%) | 0.69 | 0 ( 0 ) | 2.11 | 1.05 (46.88) | 4.12 | 2.29 (53.88) | 6.52 | 3.46 (52.03) |
| 5 (25%) | 3.13 | 0 ( 0 ) | 5.85 | 2.35 (39.30) | 8.84 | 4.57 (50.95) | 11.09 | 6.49 (53.55) |

Note: [a] $\rho_{12}$ and $\rho_{34}$ (Ratio of $\rho_{rs}$ and 900) are referred to the parameters in equation (14)
[b] the total loss of the latent travel demand, i.e. the increment of the non-revealed travel demand (NTD)
[c] The cross loss of the latent travel demand, i.e. the increment of the NTD
[d] The proportion of cross loss to total loss (%)
[e] Link capacity, $C_3$ and $C_4$ = 15 (the ratio of degraded and non-degraded capacities = 75%)

TABLE 6: The travel demand satisfaction ratio (%) from the LDE and degraded network

| Network degradation state | RTD function parameters [a] | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 0 (0%) | | 300 (33%) | | 600 (66%) | | 900 (100%) | |
| | DSI | | DSI | | DSI | | DSI | |
| | Total | No Cross | Total | No Cross | Total | No Cross | Total | No Cross |
| non-degraded | 98.84 [b] | 98.84 [c] | 96.67 | 96.67 | 93.23 | 93.23 | 88.89 | 88.89 |
| 15 (75%) [d] | 97.69 | 97.69 | 94.00 | 95.80 | 88.85 | 92.53 | 83.29 | 88.31 |
| 10 (50%) | 92.71 | 92.71 | 85.07 | 92.07 | 77.33 | 89.55 | 70.44 | 85.82 |
| 5 (25%) | 71.02 | 71.02 | 60.13 | 75.80 | 52.16 | 76.53 | 46.13 | 74.98 |

Note: [a] $\rho_{12}$ and $\rho_{34}$ (Ratio of $\rho_{rs}$ and 900) are referred to the parameters in equation (14)
[b] The DSI with total loss (i.e. with both LDE and degraded network)
[c] the DSI without cross loss (i.e. with either LDE or degraded network)
[d] Link capacity, $C_3$ and $C_4$ = 15 (the ratio of degraded and non-degraded capacities = 75%)

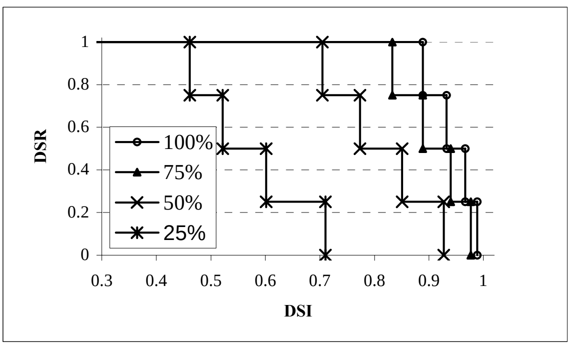



FIGURE 5: The effect of the capacity degradation on DSR

When the state of the RTD is fixed as shown in Table 6, the resultant DSR is obtained and presented in Figure 6 with corresponding to the RTD states of 0% (non-LDE), 33%, 66% and 100%. The larger is the extent of $\rho_{rs}$, the less is the DSR. For the RTD state of 100%, the DSR will be 0% as the specified DSI is more than 89%. The DSR will be 100% as the specified DSI is less than 46%. For the RTD state of 33%, the DSR will remain at 100% as the specified DSI is less than 60%. The DSR will be maintained at 0% until the specified DSI is less than 97%. It is shown that both the RTD and degraded network can influence the DSR significantly.

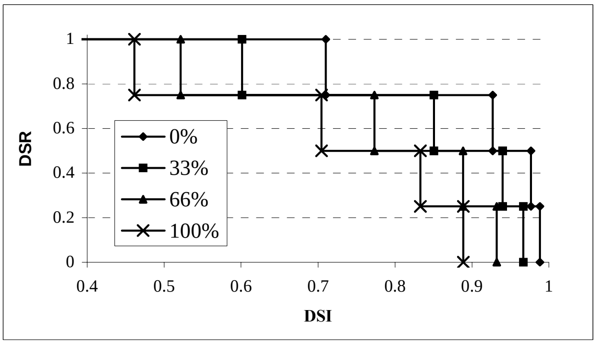



FIGURE 6: The effect of the recurrent travel demand on DSR

Furthermore, with reference to the total and no-cross loss as given in Table 6, the DSR results are compared in Figure 7. It is shown that the DSR is equal to 1.0 when the DSI is less than 0.38. The DSR is 1.0 without considering the cross loss, while the DSR is 0.77 with taking account the total loss at the DSI of 0.68. When the DSI reaches to 1.0, the DSR approaches to 0% at all. The DSR with the cross loss is always higher than that without the cross loss on the DSI. In Tables 5 and 6, it can be seen that the NTD is always under-estimated without taking into account the cross loss when both the RTD and degraded network occur simultaneously. Subsequently, the DSI and thus the DSR will be over-estimated in general.

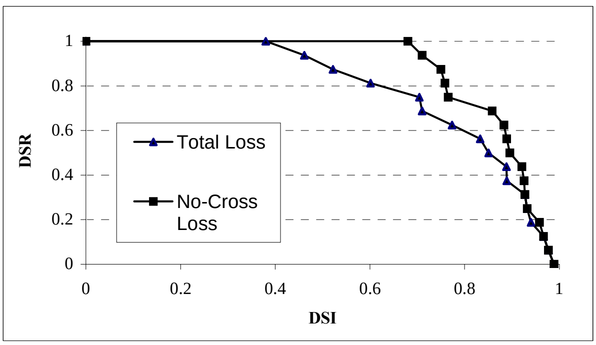



FIGURE 7: The cross effect of the recurrent travel demand and degradable capacity on DSR

On the basis of Table 4, the travel time reliability (I) can be obtained by equation (10). The results are shown in Figure 8. The travel time attains the minimum of 80.95(min). In other words, the travel time reliability (I) is 0% if the specified travel time is less than 80.95(min). The travel time reliability (I) is increasing if the specified travel time increases. In the non-degraded network state, the travel time reliability (I) approaches to 100% when the specified travel time is greater than 90(min).

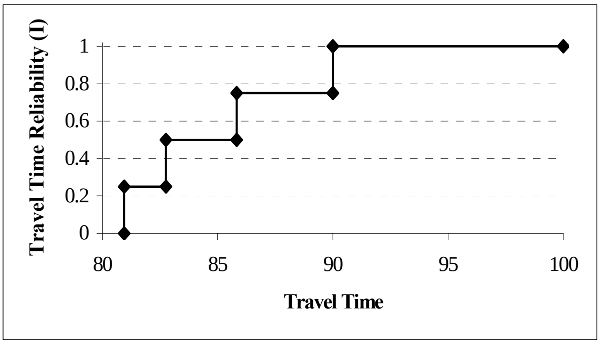



FIGURE 8: Travel time reliability (I)

Similarly, from Table 4, the travel time reliability (II) can be obtained by equations (11) and (12). The results are shown in Figure 9 with the combination of the RTD and degraded network, and the RTD states of 0%, 33%, 66% and 100% respectively. The travel time reliability (II) in the RTD state of 0% is the highest but in the RTD state of 100% is the lowest. In the RTD state of 0%, it attains 25% when the specified network service level is less than 1.01 and 100% when the specified network service level is larger than 1.39. However, in the RTD state of 100%, the travel time reliability (II) is maintained at 25% as the specified network service level is not greater than 1.06. It is remained at 100% until the specified network service level is less than 1.93.

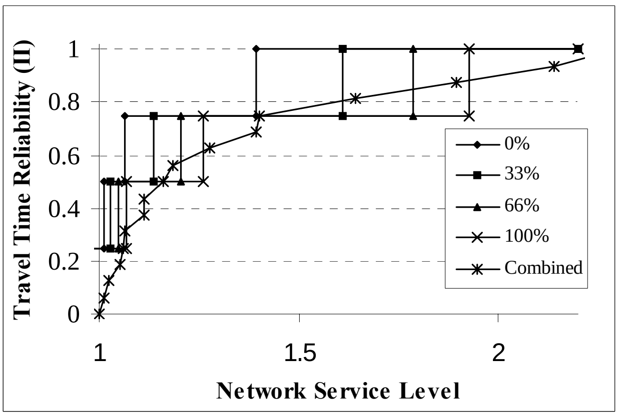



FIGURE 9: Travel time reliability (II)

From Table 4, the OD sub-system reliability can also be obtained by equation (13). Figure 10 shows the results for the RTD states of 0%, 33%, 66% and 100% respectively. The OD sub-system reliability in the RTD state of 0% is the highest but in the RTD state of 100% is the lowest. In the no- LDE state, it attains at 25% when the specified decrement rate is less than 1.1% and 100% when the required decrement rate is larger than 28.1%. However in the RTD state of 100%, it remains at 25% as the specified network service level is not more than 6.3%. It is maintained at 100% until the required network service level is less than 48.1%.

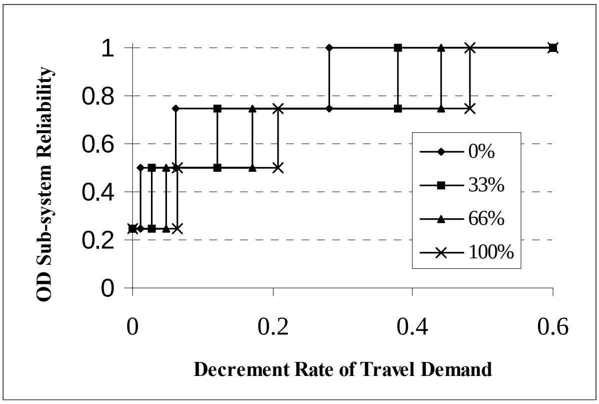



FIGURE 10: OD sub-system reliability at different latent travel demand level

It is demonstrated in the above example that the travel time and OD sub-system reliabilities are in fact the special cases of the DSR.

Let us now consider the capacity reliability with this example. Suppose that the referenced OD demand is ($q_{12}$, $q_{34}$)=(11.12, 11.12) (veh/min), which is the equilibrium OD flow when there is no LDE but with degraded network. Due to the symmetrical property of the example network, the reserved capacities are equal to (20/11.12, 15/11.12, 10/11.12, 5/11.12) in respect to the states of the network degradation given in Table.2. The resultant capacity reliability is shown in Figure 11. Chen et al. (1999) refers the reserve capacity as to travel demand level. The capacity reliability is 100% if the specified demand level is less than 0.45. When the specified demand level is greater than 0.45 and less than 0.90, the capacity reliability is 75%. The capacity reliability attains 50% as the specified demand level locates between 0.90 and 1.35. If the specified demand level is greater than 1.35 and less than 1.80, the capacity reliability would fall to 25%. The capacity reliability becomes 0% once the specified demand level is greater than 1.80.

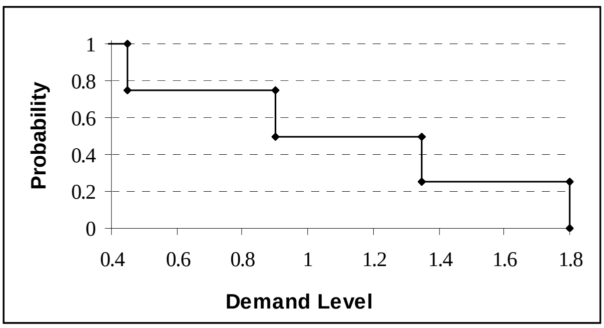



FIGURE 11: Capacity reliability with base OD demand of (11.12, 11.12)

## 6. CONCLUSIONS AND RECOMMENDATIONS

In this paper, the issues of road network reliabilities are discussed in terms of how and how much a network can satisfy the need of road users and reveal the demand of travel. In general, there are five specific indicators to measure the network performance; namely, (1) connectivity, (2) travel time by OD pair or the network service level, (3) reserve capacity (in terms of demand level), (4) decrement rate of OD flow, (5) the travel demand satisfaction ratio (DSI). The first four indicators are defined in the previous related studies. The last one, i.e. the DSI, is proposed in this paper for assessing the road network performance together with comparison against the other four indicators. The travel demand satisfaction reliability (DSR) is subsequently defined as the probability of the DSI greater than a specified value. It is shown that the travel time and system and OD sub-system reliabilities are the special cases of the DSR.

The definitions of road network reliabilities are concerned with the uncertainties — the fluctuation of traffic flow, the occurrence of network degradation and the variation of RTD. The fluctuation of traffic flow may be due to the randomness of link travel times and the variation of route choices. The link travel time is usually assumed to follow a particular probability distribution, such as normal distribution. The network degradation is concerned with the link capacity reduction. The link capacity is also assumed with a normal distribution. The randomness of the RTD results from the variation of the latent demand events (LDE). On the other hand, the definitions of the travel time, system, OD sub-system, and capacity reliabilities are mainly concerned with the uncertainty of the network degradation. However, the DSR takes into account the uncertainty of both the LDE and network degradation.

The occurrence of both the LDE and network degradation leads to more serious congestion. Thus, the cross loss of latent travel demand is greater. In other words, more latent travel demand will be suppressed. The previous related studies have not considered this cross loss. This would generally lead to the over-estimation of the travel time reliability, the system and OD sub-system reliabilities, and also cause the under-estimation of the capacity reliability.

Under the circumstances with both the network degradation and LDE, the proposed DSR can be used properly to show how and to measure how much a network can satisfy and reveal the latent travel demand. The equilibrium travel demands and latent travel demands are forecasted by an elastic travel demand function, which are dependent on the RTD states. The ratio of the ETD and LTD is defined as the DSI. The DSR is estimated by a probability function of DSI. In order to present the essential ideas of DSR, simple example networks are employed for illustration and discussion. However, the basic concept of DSR can be applicable to more complicated networks.

For future researches, the DSR will be used to evaluate the performance of alternative transport management policies, and alternative road expansion schemes with taking into account the effect of elastic demand. The concept of **customer satisfaction** (see Fornell, 1992 and Gorst et al., 1998) should be extended to network design problem for enhancing the service quality of road network. In 1994, Fornell commented that “When a customer recognises quality, it is reflected in customer satisfaction. Customer satisfaction in turn can lead to increased revenue. Customers are an economic asset. They are not on the balance sheet, but they should be.” In order to use the concept of DSR for assessing network performance, some surveys should be conducted for calibration of the elastic travel demand function and determination of the random variable distribution of the RTD. On the other hand, it may be worthwhile to extend the concept of DSR to the reliability of satisfying the activities, because the RTD is actually influenced by the activities related to the travel. Such *activity reliability* can be studied in future. In line with the relevant work by Bell (1999), an approach based on game theory can be used to identify the worst scenario (i.e. the *lower bound* of the network reliability) where travellers are extremely pessimistic about the network condition. On the other hand, the approach proposed in this paper is able to determine the *upper bound* of the network reliability (i.e. the reference case under the free flow condition). However, further research on these bounds should be justified by empirical data from the travellers' perspective.

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## APPENDIX: ANALYSIS OF NETWORK EQUILIBRIUM

In this Appendix, we consider the way in which an equilibrium assignment of traffic to the network is affected by variations in certain quantities. We consider first variations in the amount of travel between origin-destination pairs, and then consider variations in the parameters of the link cost functions. The analysis that is presented here is generic and can be applied to any form of link cost functions and variations in any of their parameters. In the context of the equilibrium between supply and demand, all of this analysis refers to the supply side. The results of this analysis can be used to estimate the way in which an equilibrium assignment will adjust in response to variations in parameters that describe network provision and state.

We start by considering the interrelationships between costs and flows on links and routes. Let

$A$ be the set of all links in the network
$v_a$ be the flow on link $a$ ,
$c_a$ be the cost of using link $a$,
$V_r$ be the flow on route $r$,

$G_r$ be the cost of using route $r$, and

$\Delta$ be the link-route incidence matrix with elements
$\Delta_{ar} = 1$ if link $a$ is on route $r$, and 0 otherwise.

Using matrix-vector notation and supposing that by default vectors are column form, the route costs are given related to link costs by

$$\mathbf{G} = \Delta'\mathbf{c},$$

and the link flows are related to the route flows by

$$\mathbf{v} = \Delta\mathbf{V},$$

where $\Delta'$ denotes the transpose of the matrix $\Delta$.

Now let

**J** be the jacobian matrix of the link cost vector with elements $J_{ab} = \partial c_a / \partial v_b$,

**M** be the jacobian matrix of the route cost vector with elements $M_{rs} = \partial G_r / \partial V_s$.

Then these jacobian matrices are interrelated as

$$\mathbf{M} = \Delta'\mathbf{J}\,\Delta.$$

We wish to establish a relationship in equilibrium between changes in the amount of travel that takes place through a network and the consequent changes in route flows and costs. Let **N** be the matrix of derivatives of route cost with respect to origin-destination flows, where $N_{r,od} = \mathrm{d}G_r / \mathrm{d}q_{od}$. We start by observing that **N** can be expressed using the chain rule as

$$\mathbf{N} = \mathbf{M}\,\mathbf{R},$$

where **R** is the matrix of derivatives of path flows with respect to origin-destination flows with elements $R_{r,od} = \mathrm{d}V_r / \mathrm{d}q_{od}$. From this, provided that the matrix **M** is invertible, we can extract the relationship

$$\mathbf{R} = \left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\mathbf{N}.$$

Now the relationship between route costs and origin-destination flows depends on the assignment rule that is used. In the present analysis, we adopt Wardrop's (1952) equilibrium principle so that following Beckmann et al. (1956), for all routes $r$ serving an origin-destination pair $od$,

$$V_r \quad \begin{cases} > 0 \quad \Rightarrow \quad G_r = G^*_{od} \\ = 0 \quad \Rightarrow \quad G_r \geq G^*_{od} \end{cases},$$

where $G^*_{od}$ is the minimum cost of travel. Differentiating the first of these cases with respect to origin-destination flow $q_{pq}$ shows that for all routes $r$ serving an origin-destination pair $od$, the assignment of variations of traffic in equilibrium leads to

$$V_r > 0 \quad \Rightarrow \quad \frac{\mathrm{d}G_r}{\mathrm{d}q_{pq}} = \frac{\mathrm{d}G^*_{od}}{\mathrm{d}q_{pq}}.$$

This can be expressed in matrix form as

$$\mathbf{N} = \Gamma\,\mathbf{K},$$

where **K** is the matrix of derivatives of minimal origin-destination cost with respect to origin-destination flow, which has elements given by $K_{od,pq} = \mathrm{d}G^*_{od} / \mathrm{d}q_{pq}$, and $\Gamma$ is the origin-destination route incidence matrix with elements, $\Gamma_{r,od} = 1$ if route $r$ serves origin-destination pair $od$, and 0 otherwise.

For an assignment to be feasible, the sum of the flows on all routes that serve an origin-destination pair should be equal to the corresponding origin-destination flow so that $\mathbf{q} = \Gamma'\mathbf{V}$, and differentiating this with respect to $\mathbf{q}$ gives that for the route flow derivatives $\mathbf{R}$ to be feasible, they must satisfy

$$\mathbf{I} = \Gamma'\mathbf{Q},$$

where $\mathbf{I}$ is an identity matrix of appropriate dimension. Eliminating $\mathbf{R}$ in favour of $\mathbf{N}$ gives

$$\mathbf{I} = \Gamma'\left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\mathbf{N}\,,$$

and then using the equilibrium relationship for $\mathbf{N}$ gives

$$\mathbf{I} = \Gamma'\left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\Gamma\,\mathbf{K}\,,$$

where $\mathbf{K}$ is the rate of change of origin-destination cost with respect to origin-destination flows. Provided that the matrix $\mathbf{K}$ is invertible, it can be expressed as

$$\mathbf{K} = \left[\Gamma'\left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\Gamma\right]^{-1}.$$

From this, an expression for $\mathbf{N}$, the rate of change of route cost with respect to origin-destination flow, is found to be

$$\mathbf{N} = \Gamma\left[\Gamma'\left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\Gamma\right]^{-1}.$$

Finally the matrix $\mathbf{R}$ of derivatives of path flows with respect to origin-destination flows is found as

$$\mathbf{R} = \left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\Gamma\left[\Gamma'\left(\Delta'\mathbf{J}\,\Delta\right)^{-1}\Gamma\right]^{-1}.$$

Together, these expressions for the matrices $\mathbf{K}$, $\mathbf{N}$ and $\mathbf{R}$ constitute the results of the sensitivity analysis of equilibrium assignment to variations in origin-destination flows.

Next, we develop a corresponding analysis for the sensitivity of an equilibrium assignment with respect to variations in the parameters $\boldsymbol{\Psi}$ of the link cost functions. Here, we denote $\mathbf{J}^{\Psi}$ as the matrix of derivatives of the vector of link costs with respect to the parameters $\boldsymbol{\Psi}$ that has elements $J_{ai}^{\Psi} = \partial c_a / \partial \Psi_i$, and $\mathbf{M}^{\Psi}$ as the matrix of derivatives of the vector of route costs with respect to the parameters $\boldsymbol{\Psi}$ that has elements $M_{ri}^{\Psi} = \partial G_r / \partial \Psi_i$.

These matrices are interrelated as $\mathbf{M}^{\Psi} = \Delta'\mathbf{J}^{\Psi}$.

If some variation is made to the link cost parameters $\boldsymbol{\Psi}$, then this will affect path costs directly and will also influence the assignments, and hence will also influence the paths costs indirectly. Let $\mathbf{R}^{\Psi}$ be the matrix of derivatives of equilibrium path flows with respect to link cost parameters with elements $R_{r,i}^{\Psi} = \mathrm{d}V_r / \mathrm{d}\Psi_i$, and $\mathbf{K}^{\Psi}$ be the matrix of derivatives of minimal origin-destination cost with respect to link cost parameters with elements given by $K_{od,i}^{\Psi} = \mathrm{d}G_{od}^{*} / \mathrm{d}\Psi_i$.

The equilibrium condition requires that the total derivative of path costs with respect to parameter values satisfies $\mathbf{M}\,\mathbf{R}^{\Psi} + \mathbf{M}^{\Psi} = \Gamma\,\mathbf{K}^{\Psi}$ so provided that $\mathbf{M}$ is invertible,

$$\mathbf{R}^{\Psi} = \mathbf{M}^{-1}\left(\Gamma\,\mathbf{K}^{\Psi} - \mathbf{M}^{\Psi}\right).$$

For assignments to remain feasible, the derivative of the origin-destination flows with respect to the parameter values should be zero, so that

$$\Gamma'\,\mathbf{R}^{\Psi} = \mathbf{0}.$$

Eliminating $\mathbf{R}^{\Psi}$ gives

$$\Gamma' \mathbf{M}^{-1}\left(\Gamma \mathbf{K}^{\Psi} - \mathbf{M}^{\Psi}\right) = \mathbf{0}\,.$$

This can be rearranged to give the expressions

$$\mathbf{K}^{\Psi} = \left(\Gamma' \mathbf{M}^{-1}\Gamma\right)^{-1}\Gamma' \mathbf{M}^{-1}\mathbf{M}^{\Psi}$$

and

$$\mathbf{R}^{\Psi} = \mathbf{M}^{-1}\left[\Gamma\left(\Gamma' \mathbf{M}^{-1}\Gamma\right)^{-1}\Gamma' \mathbf{M}^{-1} - \mathbf{I}\right]\mathbf{M}^{\Psi}\,.$$

Together, these expressions for the matrices $\mathbf{K}^{\Psi}$ and $\mathbf{R}^{\Psi}$ constitute the results of the sensitivity analysis of equilibrium assignment to variations in link cost parameters $\boldsymbol{\Psi}$. We note that this sensitivity analysis will not be applicable when the number of origin-destination pairs is larger than the number of links in the network because certain matrixes involved necessarily become non-inevitable (see Bell and Iida, 1997). However, more recently, Yang and Bell (2007) have shown that corresponding results are available irrespective of network size and topology.