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HOUSING ALLOCATION PROBLEM IN A CONTINUUM TRANSPORTATION SYSTEM

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We consider a city with a central business district (CBD) with a road network outside of the CBD that is relatively dense and is considered to be a continuum. In this transportation system, several classes of users with different perceptions and behavior are considered. Their demands are continuously distributed over the city, and their travel patterns to the CBD satisfy the user equilibrium conditions under which each individual user chooses the least costly route in the continuum to the CBD. A logit-type demand distribution function that incorporates housing rent and travel cost is specified to model the housing location choice behavior of the commuters. A bi-level model is set up for modeling the housing allocation problem in the continuum transportation system. At the lower level, a set of differential equations is constructed to describe this housing location and traffic equilibrium problem. We present a promising solution algorithm that applies the finite element method (FEM) to solve this set of differential equations. At the upper level, a constrained minimization problem is set up to find the optimal housing provision pattern that maximizes the total utility of the system. The FEM and convex combination method are proposed to solve the minimization problem with the sensitivity information from the lower level. A numerical example is given to show the workability of the proposed bi-level model and the effectiveness of the solution algorithm.

KEYWORDS: Housing allocation problem, transportation system, continuum model, bi-level programming, finite element method

1. INTRODUCTION

It is well recognized that land-use and transportation systems interact extensively with each other, and that the consideration of only one of these systems will not fully address the system's responses to major infrastructure developments and policy changes. In the past decades, there were a number of influential studies linking the land-use (especially housing) with the transportation system. One of the earliest works was Solow and Vickrey (1971), which studied the optimal allocation of land between housing and transportation in a linear city. They suggested that the market value of land was a poor guide on land-use decisions. Further to this research, Solow (1972), Mirrlees (1972), Dixit (1973), Kanemoto (1976), and Arnott (1979) introduced different models for identifying the optimal land-use pattern that maximized the per capita utility or total system utility, in a monocentric linear city with congestion-dependent transportation cost function. For a complex network system, Boyce and Matsson (1999) recently adopted the network equilibrium approach and divided the problem into two levels. At the lower level, they incorporated housing rent and home origins into the traditional discrete network equilibrium model. Housing rent is one of the main decision factors in the choice of home location. At the upper level, the social benefits that are achievable by the users of the system are maximized with respect to the allocation of housing provision, which in turn affects housing rents at the lower level. Consequently, this bi-level model provides transportation system planners with a useful tool for the optimal allocation of housing units to minimize overall travel disutility that takes into account the

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corresponding changes in land-use patterns. Based on the idea of Boyce and Matsson (1999), we propose a continuum model for the study of the macroscopic role of the effect of housing rent and the interaction between housing and travel patterns in land-use and transportation systems on a regional scale.

In the literature, the modeling of traffic equilibrium problems can be classified into two general approaches: the discrete modeling approach and the continuum modeling approach. The discrete modeling approach, in which each road link within the network is modeled separately and the demand is assumed to be concentrated at hypothetical zone centroids, is commonly adopted for detailed planning. The continuum modeling approach, in contrast, is used for the initial phase of planning and modeling in broad-scale regional studies, in which the focus is on the general trend and pattern of the distribution and travel choice of users at the macroscopic, rather than detailed, level. In the continuum approach, a dense network is approximated as a continuum in which users are free to choose their routes in a two-dimensional space. The fundamental assumption is that the differences in modeling characteristics, such as the travel cost and the demand pattern, between adjacent areas within a network are relatively small as compared to the variation over the entire network. Hence, the characteristics of a network, such as the flow intensity, demand, and travel cost, can be represented by smooth mathematical functions (Vaughan, 1987). A promising extension to modeling the study area of an arbitrary city shape and the advancement of the solution algorithm have recently been made (Wong, 1998; Wong et al., 1998; Ho and Wong, 2005a, b; Ho et al., 2005, 2006), which adopted the finite element method (FEM) to solve the resultant continuum model (Zienkiewicz and Taylor, 1989).

The continuum modeling approach has various advantages over the discrete modeling approach in macroscopic studies with very dense transportation systems (Blumenfeld, 1977; Taguchi and Iri, 1982; Sasaki et al., 1990; Gwinner, 1998). First, the problem size for modeling a dense transportation network can be significantly reduced in the continuum-modeling paradigm, because the modeled system can be approximated by a set of smooth mathematical functions that are characterized by a much smaller number of parameters. Second, less data is therefore required in setting up a continuum model, as compared to the detailed representation of a dense network in the discrete modeling approach. This makes the continuum model more appealing for macroscopic studies in the initial phase of design because the resources for the collection of data in this phase are usually very limited. Third, the continuum modeling approach offers a better understanding of the global characteristics of a road network. As the numerical results of a continuum model can be visualized in a two-dimensional sense, the influence of different model parameters and the spatial interaction between locations can easily be detected and analyzed.

In this study, the land-use system, which is characterized by housing supply and housing rent, is considered in a continuum traffic equilibrium model. Similar to the study of Boyce and Matsson (1999), this housing and transport problem is formulated as a bi-level model. The lower-level subprogram, which incorporates the housing-related parameters into the traffic equilibrium problem, is formulated as a set of differential equations (Ho and Wong, 2005b) and will be discussed in Section 2. Section 3 introduces the upper-level subprogram, which the housing provision pattern is optimized according to the total disutility incurred by users, and the solution algorithm adopted for this subprogram. Section 4 presents a numerical example to demonstrate the effectiveness of the proposed methodology.

2. LOWER-LEVEL SUBPROGRAM

2.1 Model formulation

This section introduces the lower-level subprogram for the housing and transportation problem, in which we consider a modeled city of an arbitrary shape with a single CBD, as shown in Figure 1. We assume that no users pass through the outer boundary. The inner boundary embraces the central business district (CBD) of the modeled city, in which activities such as work and school are concentrated. Users travel between their demand locations in the city region and the CBD along the least costly route. For broad-scale planning, we assume that housing is continuously distributed in the suburbs and that the activities are concentrated in the CBD. The model describes the interaction between the housing locations of users and the traffic equilibrium pattern.

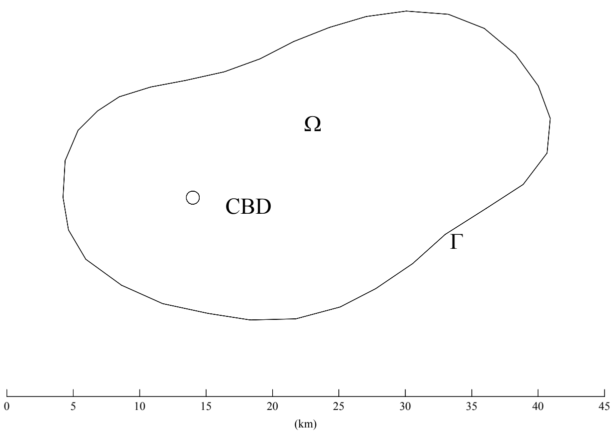


FIGURE 1: The modeled city with an arbitrary shape for the housing location model

Several classes of users are considered in this housing location model and different classes of users have different perceptions of travel costs, housing rent, and congestion costs. The incorporation of multiple classes of users adds realism to the model representation, because users with different perspectives and behavior compete for resources, such as road space and housing units. The set of differential equations that governs this housing location model comprises the flow conservation equation, travel cost potential, demand distribution function, and several boundary conditions. Consider the following flow conservation equation.

$$\nabla \cdot \mathbf{f}_m(x,y) + q_m(x,y) = 0, \quad \forall (x,y) \in \Omega, m \in N_M \,, \tag{1}$$

where $\mathbf{f}_m(x,y) = (f_{mx}(x,y), f_{my}(x,y))$ is the flow vector (expressed as the number of users that cross a unit width) of class m users at location (x,y) within the modeling region Ω; $q_m(x,y)$ is the density of demand (expressed as the number of users per unit area) of class m users at location (x,y); and N_M is the number of classes. We now define the travel cost potential. Let $c(x,y)$ be the unit travel time (that is expressed in hours per unit length of travel) at (x,y) and this unit travel time function is defined as

$$c(x,y)=a(x,y)+b(x,y)\left(\sum_{i=1}^{N_M}\left|\mathbf{f}_i(x,y)\right|\right)^{\gamma}, \quad \forall(x,y)\in\Omega, \tag{2}$$

where $a(x,y)$ is the free-flow travel time and $b(x,y)$ is the congestion sensitivity parameter at location (x,y). $\left|\mathbf{f}_m(x,y)\right|=\sqrt{f_{mx}(x,y)^2+f_{my}(x,y)^2}$ is the flow intensity of class m users. The parameter γ is used to account for the nonlinear effect of the total flow on the local unit travel time. For a positive value of time p_m the unit travel time as defined in equation (2) can be transformed into the unit transportation cost $c_m(x, y)$ (that is expressed in dollars per unit length of travel) at (x,y) for class m users, and is defined as follows.

$$c_m(x,y)=p_m c(x,y)=a_m(x,y)+b_m(x,y)\left(\sum_{i=1}^{N_M}\left|\mathbf{f}_i(x,y)\right|\right)^{\gamma}, \quad \forall(x,y)\in\Omega, m\in N_M, \tag{3}$$

where p_m is the value of time for the transformation of the unit travel time into the unit transportation cost for class m users. $a_m(x,y)$ and $b_m(x,y)$ are the free-flow and congestion-related parameters, respectively, for class m users. The different unit transportation costs of these classes of users are due to the different values that they assign to travel time and their tolerance for congestion. Furthermore, for a given flow pattern $\mathbf{f}_m$ and unit transportation cost c_m, we consider a scalar function $u_m(x,y)$, which can be proven to be the total travel cost of class m users at location (x,y), for class m users that satisfies the following relation.

$$c_m(x,y)\frac{\mathbf{f}_m(x,y)}{\left|\mathbf{f}_m(x,y)\right|}+\nabla u_m(x,y)=0, \quad \forall(x,y)\in\Omega,\ m\in N_M. \tag{4}$$

In equation (4), the flow vector is the direct opposite of the gradient of the scalar function $u_m(x,y)$, that is,

$$-\mathbf{f}_m(x,y)\,//\,\nabla u_m(x,y), \quad \forall\mathbf{f}_m\neq 0, \tag{5}$$

where "//" means that the two vectors are in the same direction. The norm of this gradient equals the local unit transportation cost, that is,

$$c_m(x,y)=\left|\nabla u_m(x,y)\right|. \tag{6}$$

For any used path l of a class m user from home location $O(x,y)\in\Omega$ to the CBD (D), the total travel cost that is incurred by the user can be obtained as

$$\bar{u}_l\left[O(x,y),D\right]=\int_l c_m\,\mathrm{d}s=\int_l c_m\frac{\mathbf{f}_m}{\left|\mathbf{f}_m\right|}\cdot\mathbf{ds}=-\int_l\nabla u_m\cdot\mathbf{ds}=-\left(u_m(D)-u_m(O)\right)=u_m(O), \tag{7}$$

using equation (4) and the fact that $\mathbf{f}_m/\left|\mathbf{f}_m\right|$ is a unit vector that is parallel to $\mathbf{ds}$ along path l. Therefore, the total travel cost is independent of the used paths, and $u_m(x,y)$ can be interpreted as the total travel cost or the cost potential that is incurred by class m users. In contrast, for any unused path $\tilde{l}$ of a class m user between the home location $O(x,y)\in\Omega$ and the CBD, the total travel cost that is incurred by the user is

$$\bar{u}_{\tilde{l}}\left[O(x,y),D\right]=\int_{\tilde{l}} c_m\,\mathrm{d}s\geq\int_{\tilde{l}} c_m\frac{\mathbf{f}_m}{\left|\mathbf{f}_m\right|}\cdot\mathbf{ds}=-\int_{\tilde{l}}\nabla u_m\cdot\mathbf{ds}=-\left(u_m(D)-u_m(O)\right)=u_m(O), \tag{8}$$

which is also based on equation (4). Thus, $\bar{u}_{\tilde{l}}[O(x,y),D] \geq u_m(O)$. The inequality in this derivation occurs because there might be some segments of the path $\tilde{l}$ on which the normal vectors $\mathbf{f}_m/|\mathbf{f}_m|$ and $\mathbf{ds}$ are not parallel, and thus $\mathrm{d}s \geq (\mathbf{f}_m/|\mathbf{f}_m|)\cdot\mathbf{ds}$. Therefore, the total travel cost for any unused paths is greater than or equal to that for the used paths. The model thus guarantees that the users of the same class choose the least costly route in the city region in a user-optimal manner.

The interaction between housing location choice and traffic equilibrium is governed by a demand distribution function, which is used to represent the way in which users choose their home location in the city. Consider the demand of class m users in an infinitesimally small area $\delta\Omega$. We assume that the choice of home location is based on the utility, U_m (which solely depends on the housing rent at that home location and the travel impedance between the home location and the CBD), that these users experience and is specified as

$$\mathrm{d}Q_m = k_m \exp\left(\omega_m U_m\left(\bar{r}_m(x,y),u_m(x,y)\right)\right)\mathrm{d}\Omega = q_m(x,y)\,\mathrm{d}\Omega\,, \tag{9}$$

where ω_m is a positive scalar parameter that measures the sensitivity of class m users to the utility level that is associated with location (x, y), k_m is the proportional constant, Q_m denotes the fixed total demand of class m users in the city, $\mathrm{d}Q_m$ is the demand that is distributed to the infinitesimally small area, and

$$U_m\left(\bar{r}_m(x,y),u_m(x,y)\right) = -\bar{r}_m(x,y) - u_m(x,y), \quad \forall (x,y)\in\Omega, m\in N_M \tag{10}$$

is the utility function that is perceived by class m users at location (x, y). This utility function consists of two components. The first term is housing rent $\bar{r}_m(x,y) = \alpha_m(x,y)\left(1+\beta_m(x,y)q(x,y)/(H(x,y)-q(x,y))\right)$, which depends on the total demand density $q(x,y)=\sum_{j=1}^{N_M} q_j(x,y)$ in this location and the total housing supply density $H(x,y)$ (expressed as the number of housing units per unit area and is held fixed in the lower-level subprogram), and where $\alpha_m(x,y)$ and $\beta_m(x,y)$ are positive scalar parameters that represent the fixed and demand-dependent components of the rent function at location (x, y). The fixed rent component represents the minimum rent that a resident has to pay to occupy a housing unit, where the demand-dependent component represents the additional rent that residents have to spend to secure a housing unit in a competitive market, in which housing rents increase with demand. As the aim of this paper is to give a preliminary study of the effect of housing provision and housing rent on demand distribution, a simple and exogenously defined housing rent function is adopted instead of considering the demand and supply of the housing market for endogenous equilibrium rent. The housing rent function is taken such that when the demand approaches the housing supply, the housing rent (or more precisely the demand-dependent term of the housing rent function) increases to a very large value. This functional form is based on the following rationales. First, when comparing the vacant housing unit to the occupied housing unit, the vacant unit normally has a higher disutility due to its poorer view, higher maintenance cost, etc. Thus, the demand-dependent term of the housing rent increases as the vacancy of housing unit decreases. Second, this functional form ensures that the demand does not exceed the housing provision, as no demand is distributed to a location at which the housing rent approaches infinity.

The second term in equation (10) is the total travel cost that class m users experience between their home location and the CBD. This form of utility function assumes that users decide upon their home location according to two criteria, housing rent and total travel cost. The first equality of equation (9) shows that the housing demand for a given area is directly proportional to the exponential of the perceived utility that users experience and the size of the area.

As the housing rent function $\bar{r}_m(x,y)$ is solely dependent on the total demand density $q(x,y)$, the utility function $U_m(\bar{r}_m(x,y), u_m(x,y))$ can then denoted as $U_m(q(x,y), u_m(x,y))$. We could then integrate the first equality of equation (9) over the whole modeled region, which produces

$$\iint_\Omega \mathrm{d}Q_m = \iint_\Omega k_m \exp(\omega_m U_m(q(x,y), u_m(x,y)))\mathrm{d}\Omega \,. \tag{11}$$

Hence, we have

$$k_m = \frac{Q_m}{\iint_\Omega \exp(\omega_m U_m(q(x,y), u_m(x,y)))\,\mathrm{d}\Omega} \,. \tag{12}$$

Substituting k_m into the second equality of equation (9) produces

$$q_m(x,y) - Q_m \frac{\exp(\omega_m U_m(q(x,y), u_m(x,y)))}{\iint_\Omega \exp(\omega_m U_m(q(x,y), u_m(x,y)))\,\mathrm{d}\Omega} = 0, \quad \forall (x,y) \in \Omega,\, m \in N_M \,. \tag{13}$$

Equations (1), (4), and (13) constitute the governing equations of this housing location model. In addition to these governing equations, the following two boundary conditions should also be satisfied.

$$u_m = 0, \quad \forall (x,y) \in \Gamma_c, \forall m \in N_M \,, \tag{14}$$

$$\mathbf{f}_m = 0, \quad \forall (x,y) \in \Gamma, \forall m \in N_M \,. \tag{15}$$

In equation (14), as users on Γ_c are already at the CBD boundary, transportation cost should not be incurred, which means that the travel cost potential vanishes. Equation (15) ensures that there is no flow across the outer boundary Γ of the modeled city.

2.2 Solution algorithm

The finite element method (FEM) is used to approximate the continuous variables in the modeled city (Zienkiewicz and Taylor, 1989). As there is no explicit objective function for this housing location model, the mixed finite element procedure that was developed by Wong et al. (1998) cannot be applied directly. Thus, we adopt the Galerkin formulation of the weighted residual technique (Cheung et al., 1996; Zienkiewicz and Taylor, 1989). In addition to the Galerkin formulation, a modified trust region method that is known as the Levenberg-Marquardt Iterative Scheme (Ho and Wong, 2005a) can be used to effectively solve least-squares problems for large and sparse systems. By using the Galerkin formulation, the differential equations (1), (4), and (13) are transformed into the following equivalent integral expressions.

$$\iint_\Omega \left(c_m(x,y) \frac{\mathbf{f}_m(x,y)}{|\mathbf{f}_m(x,y)|} + \nabla u_m(x,y) \right) \psi(x,y)\,\mathrm{d}\Omega = 0, \quad \forall m \in N_M, \psi(x,y), \tag{16}$$

$$\iint_\Omega \left(q_m(x,y) - Q_m \frac{\exp(\omega_m U_m(q(x,y), u_m(x,y)))}{\iint_\Omega \exp(\omega_m U_m(q(x,y), u_m(x,y)))\,\mathrm{d}\Omega} \right) \psi(x,y)\,\mathrm{d}\Omega = 0, \quad \forall m \in N_M, \psi(x,y), \tag{17}$$

$$\iint_\Omega (\nabla \cdot \mathbf{f}_m(x,y) + q_m(x,y))\psi(x,y)\,\mathrm{d}\Omega = 0, \quad \forall m \in N_M, \psi(x,y), \tag{18}$$

where $\psi(x,y)$ is the trial (or weight) function in the weighted residual technique. Boundary conditions (14) and (15) are enforced by taking a zero weight function (Cheung et al., 1996). In the Galerkin formulation, the local interpolation function of the finite element is used as the trial function. The modeling area is first discretized into a finite element mesh, in which the Galerkin formulation is applied at the element level. The governing equations for all user classes at a particular finite element node s are given as follows.

$$\mathbf{r}_{sm}(\mathbf{\Psi}) = \begin{Bmatrix} r_{sm1} \\ r_{sm2} \\ r_{sm3} \end{Bmatrix} = \begin{Bmatrix} \sum_{e \in E_s} \iint_{\Omega_e} \left(c_m(x,y) \dfrac{\mathbf{f}_m(x,y)}{|\mathbf{f}_m(x,y)|} + \nabla u_m(x,y) \right) N_s(x,y)\,\mathrm{d}\Omega \\ \sum_{e \in E_s} \iint_{\Omega_e} \left(q_m(x,y) - Q_m \dfrac{\exp(\omega_m U_m(q(x,y), u_m(x,y)))}{\iint_\Omega \exp(\omega_m U_m(q(x,y), u_m(x,y)))\,\mathrm{d}\Omega} \right) N_s(x,y)\,\mathrm{d}\Omega \\ \sum_{e \in E_s} \iint_{\Omega_e} (\nabla \cdot \mathbf{f}_m(x,y) + q_m(x,y)) N_s(x,y)\,\mathrm{d}\Omega \end{Bmatrix}, \tag{19}$$

where Ω_e denotes the domain of the finite element e, E_s is the set of finite elements that connects node s, $N_s(x,y)$ is the local interpolation function of the finite element that corresponds to node s, $\mathbf{r}_{sm}$ is the nodal residual vector for class m users at node s, which represents the extent to which the governing equations (1), (4), and (13) are locally satisfied around node s, and $\mathbf{\Psi}$ is the solution vector of the problem. For the global satisfaction of the governing equations, we require that

$$\mathbf{R}(\mathbf{\Psi}) = \mathrm{Col}(\mathbf{r}_{sm}(\mathbf{\Psi})) = \mathbf{0}, \tag{20}$$

which defines a system of non-linear equations. We apply the Newton-Raphson algorithm with a line search to solve the problem, in which we derive the iterative equation

$$\mathbf{\Psi}_{k+1} = \mathbf{\Psi}_k - \lambda \mathbf{J}_k^{-1} \mathbf{R}_k, \tag{21}$$

where $\mathbf{J}_k$ is the Jacobian matrix of vector $\mathbf{R}_k$ in iteration k, and λ is the step size, which is determined by the golden section method (Sheffi, 1985). The solution procedure is summarized as follows.

<u>Solution Procedure A</u>

Step A1: Find an initial solution $\mathbf{\Psi}_0$. Set $k = 0$.

Step A2: Evaluate $\mathbf{R}(\mathbf{\Psi}_k)$ and $\mathbf{J}(\mathbf{\Psi}_k)$.

Step A3: If the relative error $|\mathbf{R}(\mathbf{\Psi}_k)|/|\mathbf{\Psi}_k|$, is less than an acceptable error ε, then terminate, and $\mathbf{\Psi}_k$ is the solution.

Step A4: Otherwise, apply the golden section method to determine the step size λ^*, which minimizes the norm of the residual vector $\left|\mathbf{R}\left(\mathbf{\Psi}_k - \lambda \mathbf{J}(\mathbf{\Psi}_k)^{-1} \mathbf{R}(\mathbf{\Psi}_k)\right)\right|$. Then, set $\mathbf{\Psi}_{k+1} = \mathbf{\Psi}_k - \lambda^* \mathbf{J}(\mathbf{\Psi}_k)^{-1} \mathbf{R}(\mathbf{\Psi}_k)$.

Step A5: Replace $\mathbf{\Psi}_k$ with $\mathbf{\Psi}_{k+1}$. Set $k = k + 1$ and go to Step A2.

3. UPPER-LEVEL SUBPROGRAM

3.1 Model formulation

This section introduces the upper-level subprogram of land-use and transport problems that aims to find the optimal housing provision pattern that maximizes the total utility of users within the system for a given budgetary constraint. A constrained optimization problem of the housing provision variable is defined as follows.

$$\underset{\mathbf{h}}{\text{Maximize}} \;\; z(\mathbf{h}) = \iint_{\Omega} \sum_{m} q_m^*(x,y) U_m\left(q^*(x,y), u_m^*(x,y)\right) \mathrm{d}\Omega \tag{22a}$$

subject to

$$H_{\max}(x,y) - \left(h_0(x,y) + h(x,y)\right) \geq 0, \;\; \forall (x,y) \in \Omega\,, \tag{22b}$$

$$h(x,y) \geq 0, \;\; \forall (x,y) \in \Omega\,, \tag{22c}$$

$$B - \iint_{\Omega} P(x,y) h(x,y)\, \mathrm{d}\Omega \geq 0\,, \tag{22d}$$

$$\iint_{\Omega} h_0(x,y) + h(x,y) - q^*(x,y)\, \mathrm{d}\Omega \geq 0\,, \tag{22e}$$

where U_m is the utility function for class m users, $h_0(x,y)$ is the existing housing provision and $h(x,y)$ is the additional housing provision at a particular location (x, y). $H_{\max}(x,y)$ is the maximum possible housing development at location (x,y) that is constrained by the topography, existing transportation network, and planned land use pattern of that location. $q_m^*(x,y)$ and $u_m^*(x,y)$ are each the optimal demand and the total travel cost for class m users at location (x,y) as derived from the lower-level subprogram for a given housing supply density $H(x,y)$ (where $H(x,y) = h_0(x,y) + h(x,y)$). The superscript (*) means that the variables are the optimal solutions that satisfy all of the governing equations in the lower-level subprogram. Similarly, $q^*(x,y)$ represents the total demand that is derived from the lower-level subprogram, B is the budget that is available for building additional housing, and $P(x,y)$ is the provision cost of a housing unit at location (x,y). Constraint (22b) states that the total housing supply ($H(x,y)$) should always fall within the development capacity ($H_{\max}(x,y)$), which governs the maximum number of additional housing units that are added at a particular location. Constraint (22c) governs the non-negativity of the additional housing units because this study assumes that no demolition of existing housing units will occur. Constraint (22d) is the budgetary constraint that ensures that the total provision cost is within the available budget. Constraint (22e) ensures that there are enough housing units for all of the users within the system. By considering the functional form of this constraint, it is possible that some locations will not have enough housing units for the number of users, even if the constraint is satisfied. However, according to the definition of the utility function in equation (10), the demand at any point within the region is assured – through the lower-level subprogram – to fall within the housing supply.

3.2 Solution algorithm

As there are inequality constraints within the maximization problem that is defined in equation (22), the Lagrangian approach with a Newtonian algorithm (Wong et al., 1998; Wong, 1998) that is adopted for continuum models cannot be used to solve the model. Rather, the convex combination method (Sheffi, 1985) that is commonly used in the discrete networking approach is proposed. To apply the convex combination method for solving minimization problem (22), the continuous variables must be converted into discrete form that can be carried out through the application of the FEM (Cheung et al., 1996) to the maximization problem (22). In the FEM, constraints (22d) and (22e) are modified as

$$B-\sum_{n=1}^{N_{FN}}\left(h_n \sum_{e\in\Omega_{en}} \iint_{\Omega_e} P_{ei}N_iN_n + P_{ej}N_jN_n + P_{ek}N_kN_n \,\mathrm{d}\Omega\right)\geq 0\,, \tag{23}$$

$$\frac{1}{3}\sum_{n=1}^{N_{FN}}\Delta_n h_n + \frac{1}{3}\sum_{n=1}^{N_{FN}}\Delta_n\left(h_n^0 - q_n^*\right)\geq 0\,, \tag{24}$$

where N_{FN} is the total number of finite element nodes within the mesh, N_i represents the interpolation function of the FEM for node i within the finite element mesh, P_{ei} represents the provision cost of housing units at node i in element e, Δ_n is the area of finite elements that surrounds node n, Ω_{en} and Ω_e are, respectively, the region that connects with node n and the finite element e, and h_n, h_n^0, and q_n^* are the additional housing units, existing housing units, and the total demand at the finite element node n. Based on the convex combination method and the transformation in equations (23) and (24), a linear program for the determination of the descent direction of the maximization problem (22) can be set up as follows.

$$\underset{\mathbf{v}}{\text{Minimize}}\;\; \nabla z(\mathbf{h})\cdot\mathbf{v} \tag{25a}$$

subject to

$$H_{\max} - h_n^0 - v_n \geq 0,\;\; \forall n\in N_{FN}\,, \tag{25b}$$

$$v_n \geq 0,\;\; \forall n\in N_{FN}\,, \tag{25c}$$

$$B-\sum_{n=1}^{N_{FN}}\left(v_n \sum_{e\in\Omega_{en}} \iint_{\Omega_e} P_{ei}N_iN_n + P_{ej}N_jN_n + P_{ek}N_kN_n \,\mathrm{d}\Omega\right)\geq 0\,, \tag{25d}$$

$$\frac{1}{3}\sum_{n=1}^{N_{FN}}\Delta_n v_n + \frac{1}{3}\sum_{n=1}^{N_{FN}}\Delta_n\left(h_n^0 - q_n^*\right)\geq 0\,, \tag{25e}$$

where $\mathbf{h}=\mathrm{Col}(h_n, n=1,\ldots,N_{FN})$ and $\mathbf{v}=\mathrm{Col}(v_n, n=1,\ldots,N_{FN})$ are the current and auxiliary solution vectors of the linear program (25). To apply the convex combination method, the gradient of the objective function $z(\mathbf{h})$ of the original maximization problem must be evaluated. To find the gradient, derivatives such as $\partial q_m^*/\partial h$ and $\partial u_m^*/\partial h$ must be calculated and may be derived from a sensitivity test of the lower-level variables (f_{mx}^* , f_{my}^* , u_m^* , q_m^*) to the upper-level variables (**h**). By denoting the lower-level variables as $\mathbf{\Psi}_l^*$ and the upper level as $\mathbf{\Psi}_u$, equation (20) can be modified as

$$\mathbf{R}\left(\mathbf{\Psi}_l^*, \mathbf{\Psi}_u\right) = \mathbf{0}\,. \tag{26}$$

By taking a partial derivative of $\mathbf{\Psi}_u$ on the left-hand side of equation (26) and rearranging it, we have

$$\left[\nabla_{\mathbf{\Psi}_u} \mathbf{\Psi}_l^*\right] = \left[\mathbf{J}\left(\mathbf{\Psi}_l^*, \mathbf{\Psi}_u\right)\right]^{-1} \left[\nabla_{\mathbf{\Psi}_u} \mathbf{R}\left(\mathbf{\Psi}_l^*, \mathbf{\Psi}_u\right)\right]. \tag{27}$$

Equation (27) is the matrix of the sensitivity of the optimized lower-level variables ($\mathbf{\Psi}_l^*$) to the upper-level variables ($\mathbf{\Psi}_u$) and may be found for each solution of the lower-level subprogram. With the help of the sensitivity matrix in equation (27), the descent direction of the maximization problem (22) can be found by solving the linear program (25). The following solution procedure is adopted to solve the housing provision model in this housing and transport problem.

<u>Solution Procedure B</u>

Step B1: Set $k = 1$. Take the initial solution for the upper level to be $\mathbf{\Psi}_{u1} = \mathbf{h}_1 = \mathbf{0}$.

Step B2: With $\mathbf{\Psi}_{uk}$ (or $\mathbf{h}_k$), solve the lower-level problem that is based on solution procedure A to find the solution to the lower level $\mathbf{\Psi}_{lk}$.

Step B3: Using $\mathbf{\Psi}_{lk}$, evaluate the sensitivity matrix according to equation (27).

Step B4: Use the sensitivity matrix from the lower level to find the auxiliary vector $\mathbf{v}_k$.

Step B5: Apply the golden section method (with the smallest search interval of δ) to determine the step size $\lambda_k^* \in [0, 1]$, which maximizes the objection function $z\left(\mathbf{h}_k + \lambda_k\left(\mathbf{v}_k - \mathbf{h}_k\right)\right)$ from equation (22a). Then, set $\mathbf{d}_k = \mathbf{h}_k + \lambda_k^*\left(\mathbf{v}_k - \mathbf{h}_k\right)$.

Step B6: If $z\left(\mathbf{d}_k\right) > z\left(\mathbf{h}_k\right)$, then set $\mathbf{h}_{k+1} = \mathbf{d}_k$, $k = k + 1$ and go to Step B2; otherwise stop and $\mathbf{h}_k$ (or $\mathbf{\Psi}_{uk}$) is the solution to the upper-level subprogram, and $\mathbf{\Psi}_{lk}$ is the corresponding solution to the lower-level subprogram.

4. NUMERICAL EXAMPLE

A numerical example is given to demonstrate the characteristics of this bi-level housing allocation problem and also the effectiveness of the solution algorithms that are adopted for the upper and lower level. This example considers a city of an arbitrary shape, as shown in Figure 1. This city spans about 35 km from east to west and 25 km from north to south. Its CBD is located at the southwestern corner of the city. Figure 2 shows the finite element mesh that is used to solve the example problem.

In this example, we consider two classes of users. The total demand of the Class 1 users is 60,000 units and that of the Class 2 users is 80,000 units. We assume that each user takes up one housing unit and travels to the CBD during the morning peak hours and returns home along the reverse route during the evening peak hours. The travel cost represents the sum of the costs in both periods or can be interpreted as the cost of travel at that location. The sensitivity parameters for the housing choice functions in equation (13), ω_1 and ω_2 for Class 1 and Class 2 users are 0.0015 and 0.0020. The unit travel time function is

$$c(x,y) = 0.0125v(x,y) + 0.00001v(x,y)\left(\left|\mathbf{f}_1(x,y)\right| + \left|\mathbf{f}_2(x,y)\right|\right)^{1.2},$$

where $c(x,y)$ is measured in hours per kilometer, and $\mathbf{f}_1 = (f_{x1}, f_{y1})$ and $\mathbf{f}_2 = (f_{x2}, f_{y2})$ are flow vectors of Class 1 and Class 2 users at location (x,y). $v(x,y) = 1.10 - 0.005\sqrt{(x-14)^2 + (y-20)^2}$ is the factor that accounts for the variation in the location-dependent parameters of the local transportation cost function. The factor increases when the distance from the CBD decreases, which reveals the network characteristic that the junctions are more closely spaced nearer to the CBD. Hence, the parameters of the local transportation cost function increase. The value of time p_m for Class 1 and Class 2 users are 40 HKD/h and 60 HKD/h, respectively.

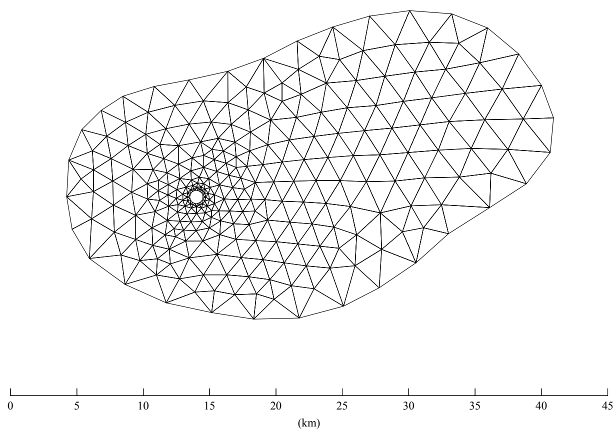


FIGURE 2: The finite element mesh for the city continuum of Figure 1

The housing rent functions are

Class 1 users: $\bar{r}_1 = 80\left(1 + 10q/(H-q)\right)$,

Class 2 users: $\bar{r}_2 = 80\left(1 + q/(H-q)\right)$,

where $\bar{r}_1$ and $\bar{r}_2$ are measured in HKD. Class 1 users are more sensitive to housing rent than Class 2 users, which means they place a greater value on housing rent when they make a decision about home location. Class 2 users, however, are more sensitive to travel cost when they choose their home location. In this bi-level model, the existing housing units, h_{0n}, is taken as a constant of 350 units/km^2 over the whole city. The maximum possible housing development is assumed to be 600 units/km^2 for all of the locations (x,y) within the modeled city. The budget that is available for additional housing units is assumed to be 1 billion HKD and the unit provision cost function that is used in this example is

$$P(x,y) = 10000\left(1.50 - 0.005\sqrt{(x-14)^2 + (y-20)^2}\right),$$

where $P(x,y)$ is measured in HKD and (14,20) is the location of the CBD. This function increases as the distance from the CBD decreases, as it is assumed that the cost of land acquisition is higher near the CBD. This assumption is reasonable because the CBD is more accessible from these locations and the land is more precious than at locations that are further away. By taking the acceptable error ε equal to 10^{-7} for the lower-level model and the smallest search interval δ as 0.02 in the upper-level model, this numerical example is solved. The convergence curves for the housing location and housing provision model are shown in Figures 3a and 3b.

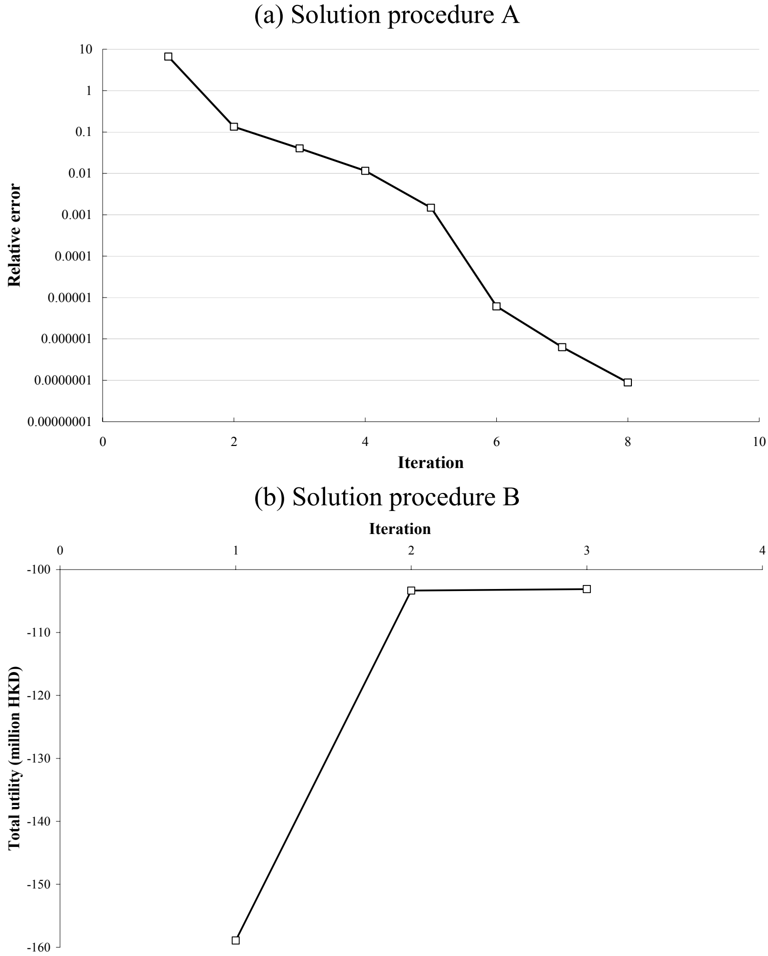


FIGURE 3: Typical convergence plots for the solution procedures

Figure 3a shows a typical convergence curve for the housing location model by using the solution algorithm A. The solution converges exponentially and meets the acceptable error in eight iterations. Figure 3b shows the convergence curve of the housing provision model. The solution to the upper-level subprogram is obtained in three iterations, which takes approximately three hours using a personal computer with P4 1.7 GHz CPU and 256 Mb RAM, when solution procedure B is employed.

Figure 4 shows the flow trajectories of system users that travel from their housing location to the CBD. These trajectories curve around the congestion areas (i.e., the proximity of the CBD) to minimize their travel cost on a user-optimal route. The total travel cost and flow intensity for Class 1 users after the upper-level optimization are shown in Figures 5 and 6. The total travel cost increases with the distance from the CBD, whereas the flow intensity decreases. The result of flow pattern, flow intensity, and total travel cost for Class 2 users are not shown in this paper as they are similar to that of Class 1 users.

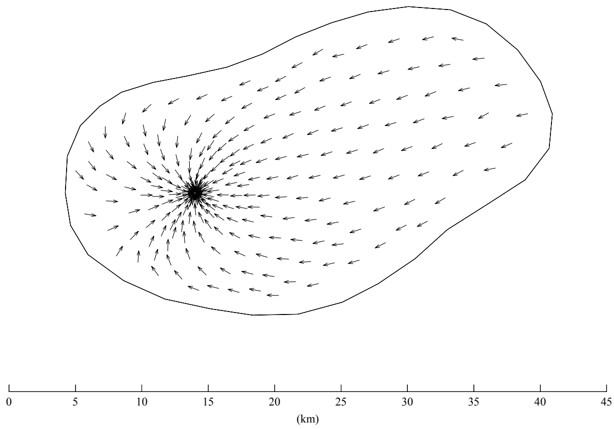


FIGURE 4: Flow pattern of Class 1 users

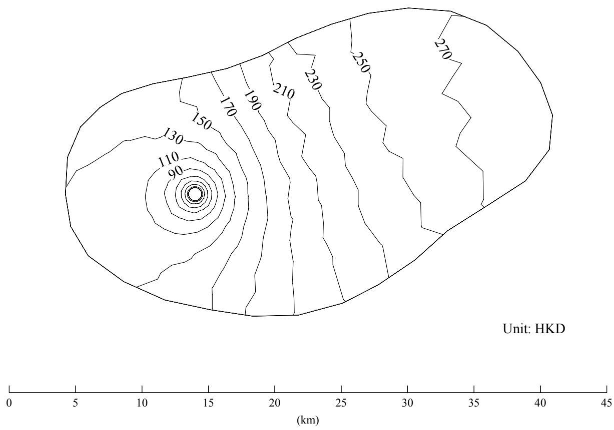


FIGURE 5: Total travel cost of Class 1 users

Figure 7 shows the additional housing units when the total utility of users within the system is maximized. The total housing density is omitted, because it has exactly the same pattern as Figure 7 but only increased by a constant value of 350 unit/km^2 that accounts for the assumed existing housing provision. In Figure 7, we can observe a fringe of optimal location for the provision of additional housing units, which is located about 10 km to the east of the CBD. According to equations (10), (22a) and the definition of the housing rent function, the total utility $z(\mathbf{h})$ depends on the demand (q_m^*), the corresponding housing rent ($\bar{r}_m$), and the total travel cost (u_m^*). Any change in the modeling environment, such as an increase in housing provision, the improvement of the transportation network, or the introduction of a housing rent subsidy, changes the housing rent and the total travel cost of a portion (or even all) of the users. Hence, the total utility of users increases or decreases based on these changes. If this fringe of high additional housing density is moved to the east, then the total utility of users within the system decreases. Under this allocation, users are encouraged to live in locations that are further away from the CBD because housing rent is lowered by the increase of supply in these locations, which in turn causes the total travel cost to increase. As this increase cannot be compensated by the additional number of users who can take advantage of the rent benefit increases due to the low housing provision cost, there is a decrease in the total utility as compared to the optimal location. In contrast, if this fringe is moved towards the CBD, then the total utility also decreases although users pay less to travel to the CBD because they prefer to live closer. This is because the number of users that are able to take advantage of the rent benefit is limited due to the high provision cost of housing at this location, which overrides the advantage of travel cost, the total utility of this additional housing allocation is smaller. By combining these two situations, it is not difficult to conclude that there is an optimal location for the additional housing units to the east of the CBD.

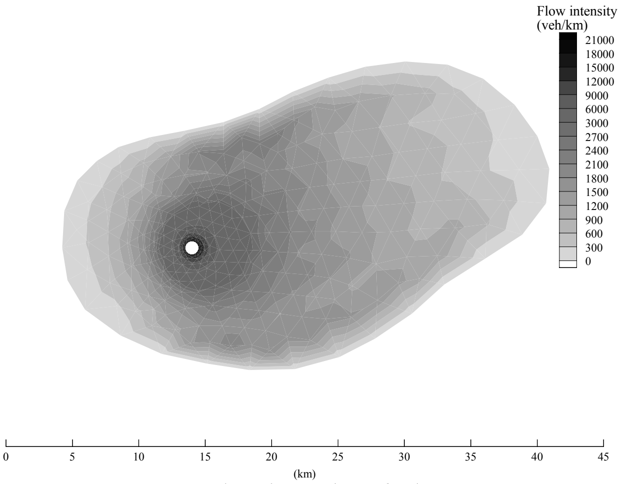


FIGURE 6: Flow intensity of Class 1 users

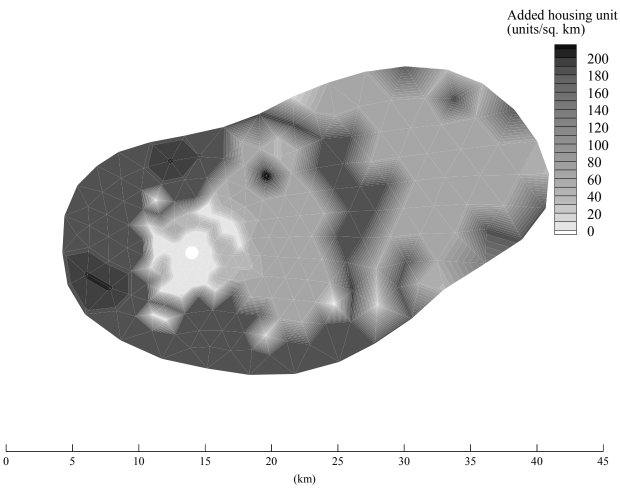


FIGURE 7: Distribution of the additional housing units

The density of the additional housing in the western part of the modeled city is also relatively high, even though the cause is slightly different from that of the optimal additional housing unit fringe on the eastern part of the modeled city. Figure 6 shows that the flow intensity is relatively low in the western part of the city, as compared with the same distance to the east of the CBD because the CBD is in the western part of the city. This causes the catchment and the flow intensity on the western side to be smaller. The allocation of housing units in the western part of the city aims to encourage users to live there and make use of the less congested traffic conditions to further increase the total utility of all of the users within the system.

Figure 8a shows the demand contours of class 1 users. The demand pattern is similar to the pattern of the additional housing units in Figure 7 because Class 1 users are more concerned about housing rent than travel cost and prefer to live in locations in which the housing rent is lower. Hence, Class 1 users are concentrated in locations with ample housing supply in which the housing rent is the lowest, and the demand pattern follows the same pattern as that of the additional housing units. Figure 8b shows the demand contours of the Class 2 users, from which it can be seen that the demand decrease with distance from the CBD because Class 2 users are more concerned about travel cost. Hence, a change in housing provision patterns does not substantially change the incentive to live close to the CBD. Figures 9a and 9b show the housing rent of Class 1 users before and after the optimization of the upper-level subprogram. At optimum, the housing rent of Class 1 users decreases as additional housing units are provided. In Figure 9b, there is a fringe of lower housing rent located about 10 km to the east of the CBD and this fringe is due to the high provision of additional housing units in these locations. The pattern of housing rent for Class 2 users after the upper-level optimization is not shown as it is similar to that of Class 1 users.

5. CONCLUSIONS

In this paper, the continuum modeling approach is extended to incorporate the interaction between land-use, which is represented by the housing supply and housing rent ideas, and transport in a single model. The land-use and transport model is formulated into a bi-level model for a city of an arbitrary shape with multiple classes of users. At the lower level, a combined housing and traffic equilibrium problem in a continuum transportation system is developed, the main objective of which is to incorporate housing-related parameters into the continuum traffic equilibrium model. This lower-level subprogram is formulated based on the Galerkin method with the weighted residual technique and the FEM. A Newtonian algorithm is adopted to solve this model, and a line search method is used to determine the step size. At the upper level, the optimal housing provision pattern is found by maximizing the total utility of the users within the system. This upper-level subprogram is formulated as a constrained maximization problem that is based on the optimized variables from the lower-level problem. By discretizing the constraints with the FEM, the convex combination method is adopted as the solution algorithm for the problem. A numerical example is given to demonstrate the effectiveness of the solution algorithms for both the upper- and lower-level subprograms, and to display the interaction between land-use and transport that is modeled by the bi-level model.

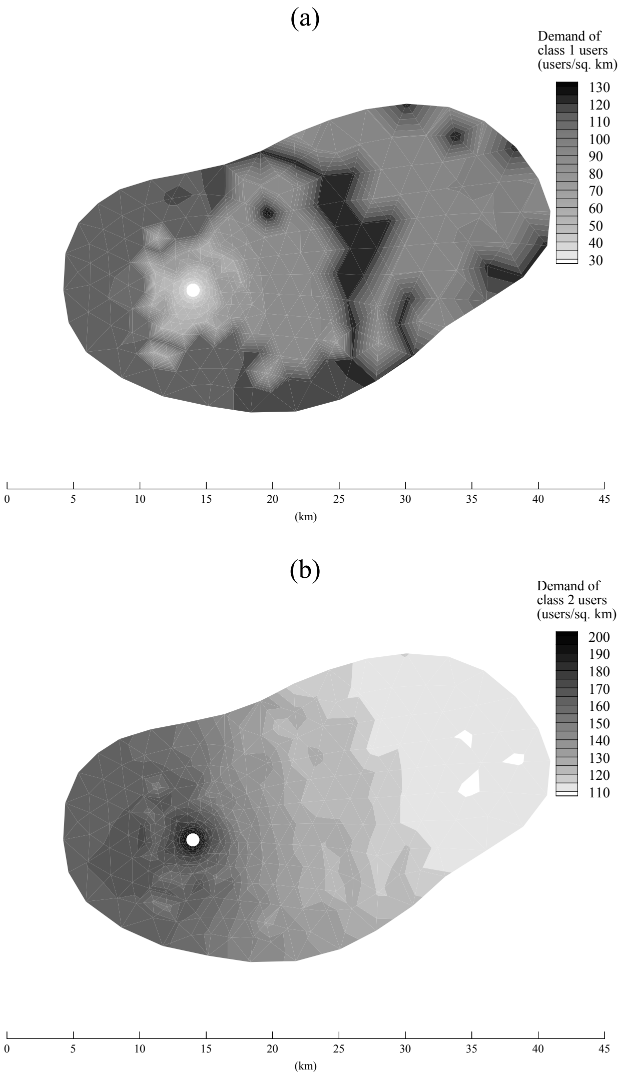
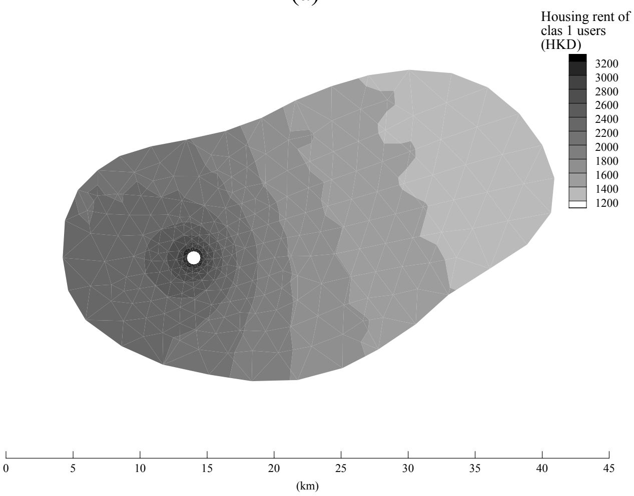


FIGURE 8: The demand contours of (a) Class 1 users; (b) Class 2 users

For this housing allocation problem there are several possible extensions for future research. First, the idea of multiple types of housing units, which differs in provision costs, rents and lot size, could be introduced. In such extension, the lower-level model could demonstrate the behavior of system users in choosing different types of housing (for example public or private housing) and their home location based on the lot size of the housing, transportation cost and housing rent. Also, the upper-level model could help the government (or the housing developers) to find the most socially beneficial (or most profitable) housing allocation pattern when competing with other housing providers. Second, by combining with the ideas introduced in Ho et al. (2006), the proposed model could be further extended to cities of multiple CBDs. Such housing allocation model with multiple CBDs could help to study the effect of housing provision on users' choice of destination. Third, the idea of congestion-pricing could be introduced into this housing allocation model. This is because the housing allocation pattern identified in this study is only a second-best solution, as the travel pattern is not socially optimized. Thus, by introducing congestion-pricing into the model, the housing allocation pattern becomes the first-best solution because both of the travel pattern and housing allocation pattern are optimized. Lastly, the idea of allocation of limited land between housing and transportation activities could also be considered for this housing allocation problem. This is because in real world land is limited, and thus the more the land is allocated to housing, the smaller amount of land can be used for other means such as transportation improvements. Therefore, a balance should be struck between the housing rent that depends on the number of housing units provided, and transportation cost that depends on the land allocated for the transportation infrastructures.

(a)



(b)

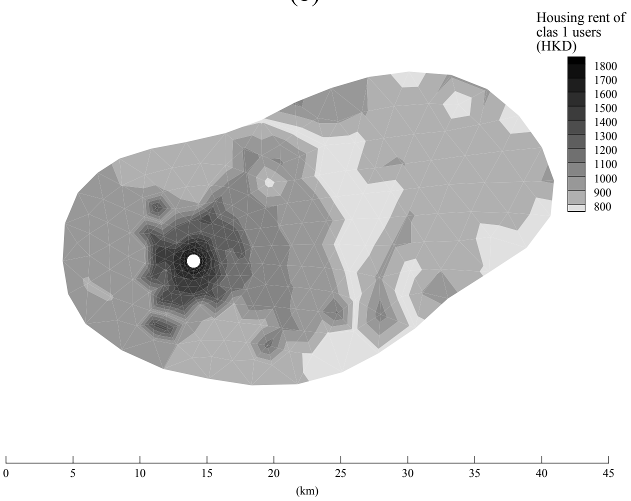


FIGURE 9: Housing rent of Class 1 users: (a) before upper-level optimization; (b) after upper-level optimization

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REFERENCES

Arnott, R. (1979) Optimal city size in a spatial economy. Journal of Urban Economics, 6, 65-89.

Blumenfeld, D.E. (1977) Modeling the joint distribution of home and workplace location in a city. Transportation Science, 11, 307-377.

Boyce, D. and Matsson, L.G. (1999) Modeling residential location choice in relation to housing location and road tolls on congested urban highway networks. Transportation Research Part B, 33, 581-591.

Cheung, Y.K., Lo, S.H. and Leung, A.Y.T. (1996) Finite Element Implementation. Blackwell Science, England.

Dixit, A. (1973) The optimum factory town. The Bell Journal of Economics and Management Science, 4, 637-651.

Gwinner, J. (1998) On continuum modeling of large dense networks in urban road traffic. In J.D. Griffiths (ed.) Mathematics in Transport Planning and Control: Proceedings of the Third IMA International Conference on Mathematics in Transport Planning and Control, Pergamon Publishing Company, New York, pp. 321-330.

Ho, H.W. and Wong, S.C. (2005a) A Levenberg-Marquardt iterative solver for least-squares problems. Communications in Numerical Methods in Engineering, 21, 327-335.

Ho, H.W. and Wong, S.C. (2005b) A combined model of housing location and traffic equilibrium problems in a continuous transportation system. Proceedings of the Sixteen International Symposium on Transportation and Traffic Theory (ISTTT16), Maryland, U.S.A., 17-21 July 2005, pp. 761-780.

Ho, H.W., Wong, S.C., Yang, H. and Loo, B.P.Y. (2005) Cordon-based congestion pricing in a continuum traffic equilibrium system. Transportation Research Part A, 39, 813-834.

Ho, H.W., Wong, S.C. and Loo, B.P.Y. (2006) Combined distribution and assignment model for a continuum traffic equilibrium problem with multiple user classes. Transportation Research Part B, 40, 633-650

Kanemoto, Y. (1976) Optimum, market and second-best land use patterns in a von Thünen city with congestion. Regional Science and Urban Economics, 6, 23-32.

Mirrlees, J.A. (1972) The optimum town. Swedish Journal of Economics, 74, 229-254.

Sasaki, T., Iida, Y and Yang, H. (1990) User-equilibrium traffic assignment by continuum approximation of network flow. Proceedings of 11th International Symposium on Transportation and Traffic Theory, Yokohama, Japan, July 1990, pp. 233-252.

Sheffi, Y. (1985) Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice-Hall, N.J.

Solow, R.M. (1972) Congestion density and the use of land in transportation. The Swedish Journal of Economics, 74, 161-173.

Solow, R.M. and Vickrey, W. (1971) Land use in a long narrow city. Journal of Economic Theory, 3, 430-447.

Taguchi, A and Iri, M. (1982) Continuum approximation to dense networks and its application to the analysis of urban road networks. Mathematical Programming Study, 20, 178-217.

Wong, S.C. (1998) Multi-commodity traffic assignment by continuum approximation of network flow with variable demand. Transportation Research Part B, 32, 567-581.

Wong, S.C., Lee, C.K. and Tong, C.O. (1998) Finite element solution for the continuum traffic equilibrium problems. International Journal for Numerical Method in Engineering, 43, 1253-1273.

Wong, S.C., Du, Y.C., Ho, H.W. and Sun, L.J. (2003) Simultaneous optimization formulation of a discrete-continuous transportation system. Transportation Research Record, 1857, 11-20.

Vaughan, R.J. (1987) Urban Spatial Traffic Patterns. Pion, London.

Zienkiewicz, O.C. and Taylor, R.L. (1989) The Finite Element Method. McGraw-Hill, International Editions.